COEFFICIENT OF VARIATION : A GEOMETRICAL EXPOSITION

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Gini's mean difference (or, Gini coefficient) and relative mean deviation can be defined geometrically by means of the Lorenz curve, which promotes our intuitive understanding of their actual meanings[1]. Coefficient of variation remains however, to be shown diagrammatically in standard textbooks. In this note, I will try to give a geometrical exposition of this summary measure.

Coefficient of variation (Mv) is defined as follows;

$$Mv^2 = \frac{1}{\mu^2}\int_0^{\bar{y}} (y-\mu)^2 f(y)\,dy,$$

where y, $\bar{y}$, μ and f(y) are a level of income, the highest level of it, the mean, and the density function, respectively. By the help of the formula of integration by parts, Mv^2 can be calculated in a following way;

$$Mv^2 = \frac{1}{\mu^2}\left[\int_0^{\bar{y}} y^2 f(y)\,dy - \mu^2\right] = \frac{1}{\mu}\left\{\left[\bar{y} - \int_0^{\bar{y}} \phi(y)\,dy\right] - \mu\right\}$$

$$= \frac{1}{\mu}\int_0^{\bar{y}} \left[F(y) - \phi(y)\right]dy,$$

where F(y) is the distribution function (or, the cumulative population ratio), and $\phi(y)$ means the proportion of total income received by the bottom x%, which is implicit in the Lorenz curve. Mathematically, we have

$$F(y) = \int_0^y f(t)\,dt,$$

$$\phi(y) = \frac{1}{\mu}\int_0^y t\,f(t)\,dt.$$

(Remark: $\phi(y) = \frac{1}{\mu}\left[y\,F(y) - \int_0^y F(t)\,dt\right]$.)

Alternatively, as can be seen, Mv^2 is homogeous of order zero with respect to y and u; namely,

$$Mv^2 = \int_0^{\bar{x}} \left[F(x) - \phi(x) \right] dx,$$

where $x = y/\mu$, and $\bar{x} = \bar{y}/\mu$.

Then, we can define Mv^2 geometrically, too. (See the Diagram I below. There, a typical "unimodal" density function is assumed.) Mv^2 is given by the area S_I.[2/]

[2/] It must be remembered that (the square of) coefficient of variation and Gini coefficient have a diagrammatical similarity. For, they are both shown by the area between F-curve and ϕ-curve, though the abscissa for the former is x (the relative income), and the one for the latter is F (the cumulative population ratio).

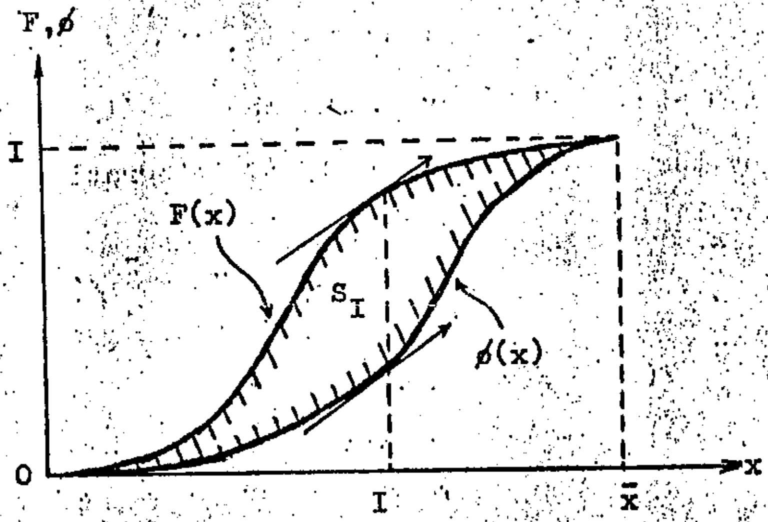

Diagram I

Footnotes:

1/ Gini coefficient can be defined geometrically as $2S_2$, and relative mean deviation as Mm in Diagram 2.

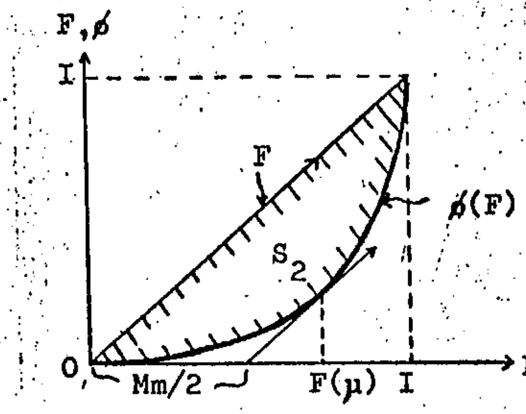

Diagram 2

2/ Mr. Yasuto Yoshizoe suggested to me that Mv^2 can be defined geometrically in quite the same way, too, in case of $\bar{y}$ infinity with the existence of variance.