The Rate of Return of Pay-As-You-Go Pension Systems
- A More Exact Consumption-Loan Model of Interest

By
Ole Settergren
Boguslaw D. Mikula
The Swedish Social Insurance Agency

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ABSTRACT

The article presents a method for calculating the cross-section internal rate of return on contributions to pension systems financed according to the pay-as-you-go principle. The method entails a procedure for valuing the contribution flow of pay-as-you-go financing, and identifies the complete set of factors that determine the cross-section internal rate of return. The procedure makes it possible to apply the algorithm of double-entry bookkeeping in analyzing and presenting the financial position and development of pay-as-you-go pension systems.

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1 INTRODUCTION

Paul Samuelson's well-known article "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money" published in 1958, has been interpreted as showing that the rate of return of pay-as-you-go pension systems, i.e. unfunded pension schemes, is the growth in the contribution, or tax, base of the system. In the absence of technological progress and with a constant number of hours worked per person, the growth in the contribution base is equal to the population growth, or Samuelson's "biological interest rate".

Several researchers have pointed out that the two-age overlapping-generation (OLG) model used by Samuelson cannot explain the dynamics of the equilibrium interest rate in a world of more than two age-overlapping generations. As Arthur and McNicoll (1978) and Willis (1988) have demonstrated, in a more than two-age overlapping-generation model, changes in the differential between the ages at which the average income is earned and consumed is a critical factor in determining equilibrium interest rates. Likewise, Keyfitz (1985, 1988), as well as Lee in numerous works (1980, 1988, 1998b, 1994a, 1994b and 2000), have shown that the amount consumed at some or all ages is affected by changes in this age differential. However, it is still surprisingly common to find statements that the rate of return on pay-as-you-go financing is equal to the growth in the contribution base.[1] Rarely are such claims accompanied by the necessary qualification that they are valid only in a two-age overlapping-generations model, or in the equally unrealistic case where both the economy and demography are in a steady state.[2]

The common assumption that the rate of return of pay-as-you-go pension systems is equal to the growth in the contribution base is rarely an efficient simplification. Recent experience in Sweden indicates how inappropriate this assumption can be. Because of the increase in life expectancy between 1980 and 2003, the pension-weighted average age of retirees increased from 72 to 75, while the income-weighted average age of contributors to the system remained relatively stable at 43, RFV (2002, 2003).[3] As a result, the differential between the average age at which contributions were paid into the system and the average age at which pensions were paid from it grew from 29 to 32 years. This 12-percent increase added 0.4 percentage point to the 0.3 real annual growth in contribution base during the period. Thus the common simplification revealed less than half of the real rate of return.

One possibly counter-intuitive effect of increases in life expectancy is consequently that they raise the rate of return for pay-as-you-go pension systems. This suggests an even more serious drawback to the simplified view than its low efficiency: its failure to reveal a

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[1] While many examples could be cited to illustrate this point, here are just two of them: "As Paul Samuelson showed 40 years ago, the real rate of return in a mature pay-as-you-go system is equal to the sum of the rate of growth in the labor force and the rate of growth in productivity." [Orszag & Stiglitz (1999) p. 15]; "The rate of return in a notional system can only be the rate of growth of the tax base that results from rising real wages and increasing numbers of employees (Samuelson 1958)" [Feldstein (2002) p. 7].

[2] In our pension context, a steady state is defined as a situation where the average wage at each age, relative to the average wage for all ages, is constant over time and where the number of retirees at each age, relative to the total number of retirees, is constant over time, i.e. where mortality rates are constant. Thus the definition of steady state is consistent with population growth (or decline) if the change rate remains constant over time.

[3] To be more precise, the average ages refer to the *expected* average ages. The expected ages will only correspond to actual average ages if fertility-driven population growth, income and mortality patterns are stable, i.e. in a steady state.

structure vital for understanding the financial dynamics of pay-as-you-go pension systems. Why then is the rate of return of pay-as-you-go financing so frequently taken to be the growth rate of the contribution base? Perhaps the answer is a belief that without the two-age OLG or steady-state assumption the analysis for determining the system's cross-section internal rate of return would be prohibitively complex.

The aim of this paper is to demonstrate that there is a clear-cut method of estimating the cross-section internal rate of return on contributions to pay-as-you-go pension systems, even when the two-age OLG and steady-state, restrictions are removed. The method entails a procedure for valuing the contribution flow of pay-as-you-go financing and identifies the complete set of factors that determine the cross-section internal rate of return. The procedure applies the algorithm of double-entry bookkeeping in analyzing and presenting the financial position and development of pay-as-you-go pension systems. These procedures are all a result of the research undertaken to reconcile certain conflicting objectives of the new Swedish pension system.[4] The method for solving, or rather managing, this problem was reached in ignorance of the above-cited research by Arthur and McNicoll, Willis, Keyfitz and Lee.[5]

In this text the phrase *cross-section* internal rate of return is used to indicate a measure distinct from the more familiar *longitudinal* internal rate of return, which is the rate of return that equates the value of the time-specific contributions from an individual or a particular group of individuals with the benefits to that individual or group. The cross-section internal rate of return is the return on the pension system's liabilities that keeps the pension system's net present value unaltered during a period of arbitrary length. However, to derive the cross-section internal rate of return, a continuous time model is used. The expression *cross-section internal rate of return* is shortened below to *rate of return*, while we sometimes use the abbreviation IRR. We also use the terms *contribution base*, *contribution rate,* and *contributions* where some would prefer *tax base*, *tax rate* and *taxes*.

Section Two presents the method for estimating the value of the contribution flow to pay-as-you-go pension systems. In Section Three this method is used to obtain a formula for calculating the rate of return on contributions to such systems, and the use of double-entry bookkeeping is sketched. In Section Four we comment on the results. In Appendix A the method for determining the value of the contribution flow, the definition of the rate of return, the and double-entry bookkeeping are illustrated by means of some numerical examples. Some readers will probably find it helpful to read the numerical examples before Sections Two and Three.

2 THE VALUE OF THE CONTRIBUTION FLOW

[4] See references for *The legislative history of the indexation and automatic balance mechanism of the Swedish pension system,* and Settergren (2000, 2003).

[5]This ignorance is clear from the legislative history of the Swedish pension reform as well as from Settergren (2000). It is evident that we were not alone in being unaware of the studies, or of their implications, that "explore the interface of richer demographic models and the overlapping-generation models of economists" (Lee, 1994a). An example is Salvador-Valdes Prieto (2000), where changes in income and mortality pattern are observed to influence the financial balance of a so-called notional defined-contribution pay-as-you-go pension scheme. However, the results are not justified by the effects that changes in income and mortality patterns have on the money-weighted age differential between the average ages when income is earned and consumed.

Pay-as-you-go financing implies that the flow of future contributions is used to finance an already accrued pension liability.[6] It is probably a matter of personal preference whether one considers that a pay-as-you-go system, by definition, has a deficit equal to this liability, or whether one accepts that its net present value is zero if contributions match pension payments. Here, the latter view is taken and financial balance is defined as,

$$\textit{Assets} - \textit{Liabilities} = 0. \tag{1.a}$$

This standard definition of financial balance is unconventional for pay-as-you-go pension systems. The usual projections of cash flows to and from pay-as-you-go pension systems for evaluating their financial situation have not traditionally been presented in the form of assets and liabilities as the methods used do not allow this to be done.[7] As already indicated, it seems reasonable to consider that a pay-as-you-go pension system whose contributions and benefits match have a zero net present value and consequently to conclude that its liability is matched by an implicit asset, referred to below as the *contribution asset*. In another context, Lee (1994 and later) uses the term *transfer wealth* for a corresponding concept.

Often pay-as-you-go systems are considered as defined by the absence of any funded assets in practice, however, there is normally a transaction account, and sometimes there are substantial funded assets. Systems without any funded assets are only a special case of the general description that follows. Hence Eq 1.a can be re-expressed as

$$CA(t) + F(t) - PL(t) = 0. \tag{1.b}$$

where

CA = contribution asset
F = buffer fund
PL = pension liability

In steady state contributions will equal pension benefits, thus $CA(t_{ss}) = PL(t_{ss})$, and $F(t_{ss}) = 0$. For each income and mortality pattern and set of pension system rules there is a unique value for the pension liability. Below Eq. 2-4 give an expression for this value in steady state.

In the case of a stable population, i.e. a population with constant mortality rates and constant population growth, the age distribution of the population can be expressed as:

$$N(x) = N(0) \cdot l\left(x\right) \cdot e^{-\gamma x}, \tag{2}$$

[6] Pension liability is defined in Eq. 3 as the present value of future benefits to all persons to whom the system has a liability at the time of valuation, minus the present value of future contributions by the same individuals. This is the *net* pension liability, however we shorten the expression to *pension liability*. The pension liability is sometimes also referred to as the implicit pension liability; see Iyer (1999). The practical problems of measuring the accrued pension liability, as well as the pension liability according to other definitions, are often substantial. Depending on the system design, the quality and the availability of data, the estimate of the pension liability may be so uncertain as to be practically useless. This paper does not deal with these important practical obstacles to employ the method suggested for estimating IRR, and the use of double-entry bookkeeping.

[7] An example of the traditional analysis of the financial status in a pay-as-you-go pension system is the Annual Report of the Board of Trustees in the Federal Old-age and Survivors Insurance and Disability Insurance Trust Funds (2003).

where

$N(x)$	= number of persons of age x
x	= age
γ	= the rate of fertility-driven[8] population growth
$l(x)$	= life-table survival function

In the system outlined, the indexation of benefits can have any relation to the average wage growth; thus, the pension benefit may vary in size relative to this average wage at different ages. If, for example, pensions are indexed by the change in consumer prices, and average wages grow at a faster rate, the average pension benefit per birth cohort will be lower for older cohorts relative to younger ones. The distribution of pensions within a cohort is ignored since it has no relevance for the cross-section rate of return.

The pension liability, V, is defined as the present value of future pension benefits to all persons to whom the system has a liability at the time of evaluation, minus the present value of future contributions by the same individuals,

$$V = \int_0^m population(x) \int_x^m PV\left[pensions(u) - contributions(u)\right] du\, dx \qquad (3)$$

where

m	= maximum age
x, u	= age

Discounting payments to and from the pension scheme by the growth in the contribution base the pension liability can be re-expressed as:

$$V = \int_0^m \underbrace{N(0)\cdot l(x)\cdot e^{-\gamma\cdot x}}_{\text{population, age } x} \int_x^m \underbrace{\frac{l(u)}{l(x)}}_{\text{survivor rate}} \cdot \underbrace{e^{-\gamma\cdot(u-x)}}_{\text{discounting}} \cdot \left[\underbrace{\overbrace{k\cdot\overline{W}\cdot e^{\varphi\cdot u}}^{\text{average pension, age u}} \cdot R(u)}_{\text{pension payments}} - \underbrace{c\cdot\overline{W}\cdot W(u)}_{\text{contributions}} \right] du\, dx, \qquad (4)$$

where

$W(x)$	= wage pattern, i.e. the average wage for age group x, as a ratio of the average wage for all age groups
$\overline{W}$	= average wage in monetary units per unit of time
c	= required contribution for a financially stable pay-as-you-go pension system
φ	= the rate of pension indexation relative to the rate of growth in the average wage
$R(x)$	= number of retirees in proportion to the number of individuals in age group x
k	= constant determining the pension level (equals the replacement rate if $\varphi = 0$)

The rate of discount is the product of the growth in the average wage times the rate of population growth. As both wages and benefits increase with the growth in the average wage, the latter cancels out of the equation, leaving the population-growth rate as the effective discount rate, γ. It would be inappropriate to use a market rate of return on capital

[8] The expression could be extended to incorporate the effects of migration on the expected contribution-weighted average age of contributors. See Settergren and Mikula (2001) for such an extended interpretation of γ.

as a discount rate. The return on capital has no impact on the financial balance of a pay-as-you-go pension system, disregarding its effect on the buffer fund if there is one.

For a stable population with stable income patterns, the contributions, C, are generated by the size of the population by age $N(x)$, the wage pattern W(x), the average wage, $\overline{W}$, and the required contribution rate for a financially stable system, c.

$$C = \int_0^m N(x)\cdot c\cdot \overline{W}\cdot W(x)\ dx. \tag{5}$$

In a steady state, the contribution rate that satisfies the financial-stability criteria of Eq. 1.a is also the contribution rate that equates pension payments in every period. Thus c is calculated as

$$\underbrace{\int_0^m N(0)\cdot l(x)\cdot e^{-\gamma\cdot x}\cdot k\cdot \overline{W}\cdot e^{\varphi\cdot u}\cdot R(x)\,dx}_{\text{pension payments}} = \underbrace{\int_0^m N(0)\cdot l(x)\cdot e^{-\gamma\cdot x}\cdot c\cdot \overline{W}\cdot W(x)\,dx}_{\text{contributions}},$$

$$c = k\cdot\frac{\int_0^m e^{-(\gamma-\varphi)\cdot x}\cdot l(x)\cdot R(x)\,dx}{\int_0^m e^{-\gamma\cdot x}\cdot l(x)\cdot W(x)\,dx}. \tag{6}$$

It is possible to obtain a measure of the pension liability in a steady state which is independent of both of the size of the contribution base and the contribution rate by dividing the pension liability by contributions paid per time unit. Thus Eq. 4 is divided by Eq. 5, where Eq. 6 is substituted for c. Rearranged and integrated by parts, this simplifies[9] to

$$\frac{V}{C} = \underbrace{\frac{\int_0^m x\cdot\left[e^{-(\gamma-\varphi)\cdot x}\cdot l(x)\cdot R(x)\right]dx}{\int_0^m \left[e^{-(\gamma-\varphi)\cdot x}\cdot l(x)\cdot R(x)\right]dx}}_{\text{average age of retirees}} - \underbrace{\frac{\int_0^m x\cdot\left[e^{-\gamma\cdot x}\cdot l(x)\cdot W(x)\right]dx}{\int_0^m \left[e^{-\gamma\cdot x}\cdot l(x)\cdot W(x)\right]dx}}_{\text{average age of contributors}}, \tag{7}$$

Eq. 7 express the fact, which may appear intuitively reasonable, that in a steady state the liability divided by contributions is equal to the differential between the average age of retirees (the first term of the RHS) and average age of contributors (the second term of the RHS). The fact that both ages are money-weighted, however, is not evident from the expression; since the average wage is a part of contributions, C, Eq. 7 is left with only the age patterns. The age differential between the average contributor and the average retiree is a measure of the duration of the pension liability, which we refer to as turnover duration, *TD*.

[9] See Appendix B for these intermediate steps.

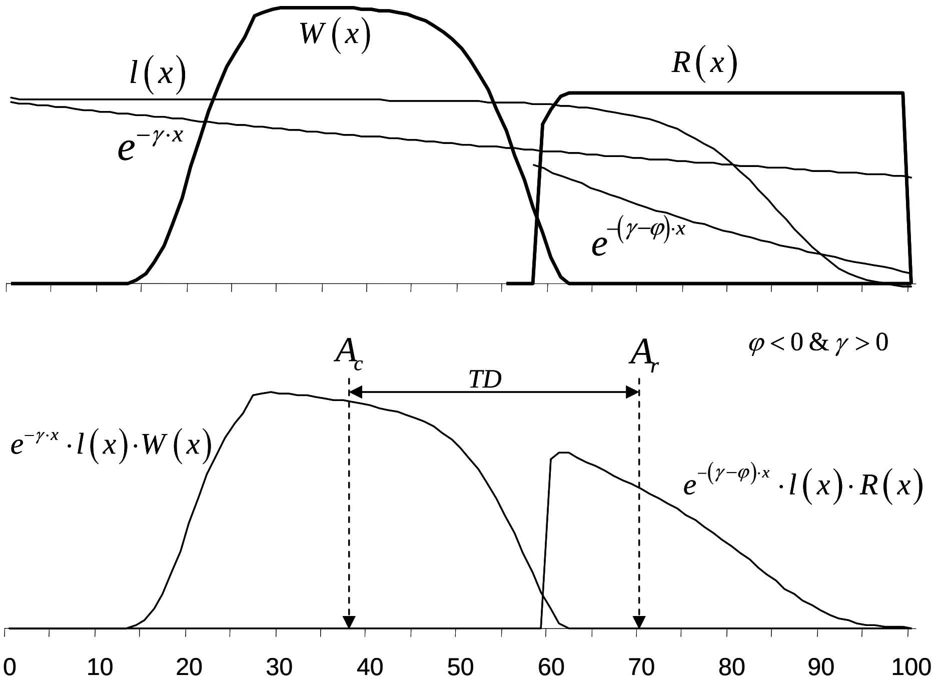
$$\frac{V}{C} = A_r - A_c = TD \tag{8}$$

where

A_r = money-weighted average age of retiree

A_c = money-weighted average age of contributor

Diagram 1. Illustration of Equations 7 and 8



Hence, for a stable population with stable income patterns, the pension liability can be separated into a volume component – contributions – and a structural component – turnover duration. Turnover duration is a useful concept: it sums up the factors that determine the scaleless, i.e. disregarding the amount of the contribution rate and the size of the contribution base, extent of the pension liability. The present value of the pension liability for a stable population with stable income patterns is expressed in years of contributions:

$$\frac{V}{C} = TD \quad \Leftrightarrow \quad V = TD \cdot C\,. \tag{9.a}$$

The separation into volume and structural components also has a temporal aspect. Except in a steady state, there will be no definite value for turnover duration; however, the current economic and demographic patterns can be used to measure the *expected* turnover duration. It is expected in the same sense as the common measure of life expectancy; i.e., it uses current observations to calculate a value which will turn out correct *ex post* only if observed patterns remain constant. The probability that any generation will live according to any published life table is virtually zero. Nevertheless, life tables are relevant and useful. Repeated estimations of expected turnover duration will reflect the changes in the financially relevant patterns and thus yield new estimates of the contribution asset, which are infinitely unlikely to produce the *ex post* correct figure. This procedure of repetitive revaluation of the contribution asset is not so different from the recurring re-evaluation of funded assets by the market.[10] For these reasons we find it appropriate to define the value of the contribution flow as the current turnover duration times the current contributions.

$$CA(t) = TD(t) \cdot C(t) \qquad (9.b)$$

Alternative definitions are possible; the reasonable ones will lead to only slightly different trajectories of the rate of return since they have to deal with the same structural components.

The expected turnover duration indicates the size of the pension liability that the present contribution flow can finance, given the present income and mortality patterns and the population-growth rate. As economic and demographic patterns change, the new value of the contribution flow can be estimated. The inverse of the expected turnover duration is a computable discount rate for the contribution flow, a measure of the current internal time preference of the pay-as-you-go pension scheme. This time preference is a function of the system design with respect to the rules that govern the indexing of pensions, the income and mortality patterns of the insured population, and the population-growth trend. Appendix C provides rough estimates of the expected turnover duration for 41 countries from Settergren & Mikula (2001). The country-specific turnover duration varies from 31 to 35 years; thus, with the internal time preferences of the hypothetical pension system in the estimate the discount rates for contributions vary between approximately 2.8 and 3.2 percent. These rates are interestingly close to the frequently assumed real interest rate of about 3 percent.

The usefulness of the expected turnover duration for valuing the contribution flow is critically dependent on the volatility of this measure. In many countries, perhaps most, the volatility of expected turnover duration can be anticipated to be moderate to low. The stem-and-leaf exhibit in Diagram 2 presents estimates of the annual percentage change in the turnover duration in Sweden in the period 1981-2003. The average increase was 0.4 percent, most of it attributable to the increase in life expectancy, the average, money-weighted age of contributors varied closely around age 43, with no clear trend. The maximum one-year increase in turnover duration was 2.1 percent; the maximum annual decrease was 0.5 percent. Over half, 12, of the annual changes were between zero and 0.5 percent, and the standard deviation of the 23 observations was 0.6.

Diagram 2. Turnover duration in Sweden, 23 annual changes, percentages 1981-2003

```
  2 | 1
  1 | 8
  1 | 11
  0 | 865
  0 | 444322000000
- 0 | 234
- 0 | 5
```

The stem-and-leaf diagram is read as follows:

```
  1 | 11  = 1.1 % and 1.1 %
  0 | 865 = 0.8 %, 0.6 % and 0.5 %
Etc |
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[10] An obvious difference is that funded assets are tradable, which make their prices much less “implicit;” however, their valuation is inevitably hypothetical to some degree as long as they are not sold off.

Source: RFV (2003, 2004)

In Section 3 the method described above for estimating the value of the contribution flow is used to derive an expression for the rate of return of pay-as-you-go pension systems and the use of double-entry bookkeeping for such systems is outlined.

3 THE RATE OF RETURN IN A PAY-AS-YOU-GO SYSTEM

Financial balance can be assured by changing either the rules of the system so that the size of the pension liability, that is the value of present and or future benefits, are adjusted or the contribution rate, i.e. the size of the contribution asset as defined by Eq. 9.b, or by doing both. Irrespective of the type of tuning employed, the financial-balance requirement of Eq. 1.b, – a net present value of zero – applies. To continue the derivation of the rate of return Eq. 1.b is rephrased as

$$TD \cdot C + F - PL = 0. \tag{10}$$

Eq.10 implies that both negative and positive funded assets are allowed and in some situations necessary in order to comply with the requirement of financial balance as defined.[11] Some of the numerical examples in Appendix A illustrate this point.

The rate of return of the pension liability that yields a net present value of zero is by definition the rate of return on contributions to the system. The formula for the rate of return of a pay-as-you-go pension system follows from differentiating Eq. 10 with respect to time,

$$\frac{d\,(TD \cdot C + F - PL)}{dt} = TD \cdot \frac{d\,C}{dt} + \frac{d\,TD}{dt} \cdot C + \frac{d\,F}{dt} - \frac{d\,PL}{dt} = 0. \tag{11}$$

The change in the pension liability is a function of the rate of return of the liability and of the difference between payments of contributions and disbursements of pensions.

$$\frac{d\,PL}{dt} = PL \cdot IRR + (C - P), \tag{12}$$

where
IRR = internal rate of return,
P = pension payments in monetary units per unit of time.

The rate of return (IRR) is both *implicit* and *explicit*. The *implicit* rate of return is a function of the impact of changes in mortality on the pension liability, and of any divergence between new pension obligations and contributions paid. In addition, changes in the rules of the system will normally alter the value of the pension liability, producing an implicit effect on the IRR. The *explicit* rate of return is the result of any explicit rules for indexing the liability, i.e. the benefits to present and future retirees.

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[11] To get a zero buffer fund in steady state the steady state rate of return on the buffer fund, or the interest rate paid on a deficit, must equal the growth in the contribution base or the valuation of the fund must reflect an assumption of a return on capital different than the growth in the contribution base.

The net difference in payments to and from the pension system is captured by the buffer fund, if there is one. Additionally, the value of the fund is changed by the return on its assets.

$$\frac{d\,F}{dt} = F \cdot r + (C - P), \tag{13}$$

where

r = rate of return on the buffer fund.

Depending on its sign and magnitude, the return on the buffer fund may increase or decrease the rate of return of a pay-as-you-go pension system. Eq. 11 can then be re-expressed as

$$TD \cdot \frac{d\,C}{dt} + \frac{d\,TD}{dt} \cdot C + F \cdot r - PL \cdot IRR = 0\,. \tag{14}$$

Finally the internal rate of return, separated into its components, is

$$IRR = \underbrace{\frac{TD \cdot \frac{d\,C}{dt}}{PL}}_{i} + \underbrace{\frac{\frac{d\,TD}{dt} \cdot C}{PL}}_{ii} + \underbrace{\frac{F \cdot r}{PL}}_{iii}\,. \tag{15}$$

Thus, the rate of return of a pay-as-you-go system is a function of:

i	*Changes in contributions*	This component consists of Samuelson's biological interest rate, changes in labour-force participation, average wage growth and changes in the contribution rate.
ii	*Changes in expected turnover duration*	This component consists of changes in income and mortality patterns and in the fertility-driven growth rate of the population.[12]
iii	*Buffer fund return*	This component consists of the return on any liquidity in the system.

The portion of the IRR resulting from mortality changes and any divergence between new pension obligations and contributions paid, or changes in system rules, can be considered as an implicit indexation of the pension liability. The IRR reduced by the rate of implicit revaluation is the rate of available indexation of the pension liability; thus

rate of available indexation = *i + ii + iii – rate of implicit indexation.* (16)

[12] Note that since the turnover duration is affected, however mildly, by changes in the fertility driven growth rate, γ, the IRR may differ from the contribution base growth even in the unrealistic case of constant mortality and income patterns. This highlights the shortcomings of the two-age OLG model, it can not represent the relevant geometry of the problem.

In practice, the rules for indexation, or the adjustment of the contribution rate or other system rules, do not necessarily distribute all the indexation available in each period of time, thus the indexation applied in any particular period can differ from what is then available. The difference is the net income or loss of the system during the period in question. The accrued value of such net income or loss is equal to the opening surplus or deficit for the next period.

rate of available indexation – rate of explicit indexation = system net income. (17)

The cross-section internal rate of return of pay-as-you-go pension systems as defined in Eq. 15 does of course have defining implications for the longitudinal internal rate of return on contributions for an individual or group of individuals.[13] But the implications are complex. For an individual the rate of return on contributions is possible to determine at the time of death; for a birth cohort, it can be settled when everyone in the cohort has died; for the pension system, when it has been closed down. Such delays in the provision of information are indeed impractical. Both participants and policymakers demand regular information on the financial position and development of the pension system. In order to produce such information, it is necessary to calculate the cross-section rate of return, on which there is only imperfect information. In the business world the problem is similar: the true rate of return can only be determined when all payments to and from a business entity have been made. As business stakeholders cannot accept such delays, accounting principles have been developed to estimate periodic rates of return for an on-going business. Since accounting measures of the rate of return, i.e. basically business net income, are arbitrary to some degree, the preferable method of determining the rate of return is a subject for debate.

For pay-as-you-go pension systems, it is possible to envisage other accounting procedures than the one described here, and other measures will normally yield a different rate of return for a specific period. By our method, the contribution flow is valued according to the expected turnover duration with cross-section observations at the time, while the pension liability is estimated with an actuarial projection that may or may not imply changes in future turnover duration. Such differences will have an impact on the trajectory of the measured rate of return, but not on the aggregate rate of return as the system approaches a hypothetical steady state.

4 CONCLUSIONS

The rate of return on contributions to pay-as-you-go pension systems is not only a function of the growth in the contribution base of the system; it is also a function of changes in income and mortality patterns and in the trend of population growth. These three factors cause changes in the average age at which contributions are paid and pensions received, i.e. changes in what we call *expected turnover duration*. Further, if there is a buffer fund in the system, the return on that fund will influence the rate of return on contributions. The rate of return can be implicitly distributed through the effects of mortality changes on the pension liability and also by differences between contributions paid and new pension liabilities. The net of the rate of return and the implicitly distributed return is the rate of financially available indexation, i.e. the explicit indexation of the pension liability which is necessary to keep the net present value of the system unaltered.

[13] The relationship can also be expressed the other way, the longitudinal IRR defines, in a complex way, the cross-section IRR.

The expected turnover duration provides an estimate of the discount rate for the contribution flow to systems with a zero pre-funding requirement for financing their obligations, i.e. pay-as-you-go systems. This makes it possible to apply a form of double-entry bookkeeping in these schemes. By means of the double-entry algorithm, the financial position of these schemes can be reported in a balance sheet, as summarized in Eq. 10, and changes in the financial position can be reported in an income statement, summarized in Eq. 17.[14] We would argue that applying the double-entry bookkeeping to pay-as-you-go pensions can improve the quality and transparency, and thus the understandability, of financial information on these important transaction systems relative to the different measures of actuarial balance used today. Disentangling the components of the rate of return also adds options for the design of pay-as-you-go pension systems; in particular, the forms of indexing pensions can be made more efficient.[15]

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[14] In practice Eq. 17 should be extended to accommodate the possibility of an opening surplus or deficit, i.e. a difference between assets (buffer fund assets and contribution asset) and liabilities.

[15] Whether double-entry bookkeeping in fact provides better information than traditional measures of actuarial balance can, perhaps, be judged from the Annual Reports of the Swedish Pension System, which have been published annually beginning with the year 2001. To judge from the index chosen for the new Swedish public pay-as-you-go pension system, separating the components of the internal rate of return adds new options for designing the indexation of pay-as-you-go systems; see *The legislative history of the income index and the automatic balance mechanism*, RFV (2001, 2002, 2003) and Settergren (2000, 2003).

APPENDIX A. NUMERICAL ILLUSTRATIONS IN AN OVERLAPPING GENERATION MODEL

A three-age overlapping-generation model is used to illustrate the impact of changes in the average ages at which income is earned and consumed. Three is the minimum number of ages needed for changing the differential between the ages at which the average income is earned and consumed. This age differential is called turnover duration, TD, formally derived in Section 2. To demonstrate the effects of changes in mortality on the rate of return, the model is extended from three to four ages.

In the model, the life of an individual is divided into three (four) periods of equal length. All individuals work for exactly two periods, at ages 1 and 2, and they are all retired for the entire third (and fourth) period, age 3 (and 4). All are born on the first day of each period; all birth cohorts are of equal size; there is no fertility-driven population growth, no migration, and no pre-retirement mortality, and everyone in retirement dies on the last day of her/his final period. There is no technological progress. Under these assumptions, the contribution base for the pension system is constant. All financial transactions are made at the end of each period. To avoid the complication that changes in contribution rate impact IRR,[16] the contribution rate is kept fixed – at 25 percent – for every period in all examples.

The effects on IRR from shifts in income and mortality patterns are described for certain alternative pension-system rules. The reason for changing system rules is to illustrate that:

- the system's cross-section IRR is independent of system design,
- the distribution of the IRR over cohorts, the "longitudinal IRR",[17] depend of system design, and
- the timing of cash flows depends on system design, even when designs are equally financially stable in the sense that they all produce a zero net present value as defined in Eq. 10.

Although the numerical examples are straightforward, the somewhat complex feature of overlapping generations, in combination with the detailed account of the effects of the shifts in income and mortality patterns, may make it tedious to work through the examples. However, this effort can be well invested as the examples, once grasped, clearly reveal structures that are vital for understanding important aspects of pay-as-you-go financing.

Example 1. A Shift in Income Pattern

Summary of what the example illustrates. In example 1 the income pattern shifts - the older workers income increases relative to the younger workers - so that the average age of contributors increases. It is shown how this decreases the turnover duration and causes a negative IRR. The effects of the negative IRR is illustrated for a pension system with rules that – exposed to this specific chock – are such that the *rate of implicit indexation* equals the negative IRR. In example 1.1 the effects of the same shift in income pattern is illustrated for a system with rules that – exposed to this specific chock – are such that the rate of implicit indexation is zero. Thus in example 1.1, to maintain a zero net present value, the negative IRR must be distributed by an *explicit* indexation equal to the IRR. The

[16] See Eq. 15 Section 3.

[17] See Section 1. Introduction for a definition of *cross-section* and *longitudinal* internal rate of return.

subsequent effects on the systems cash flows, buffer fund, etc are illustrated by means of an income statement and balance sheet.

The shift in income pattern. Up until and including Period One, the wage is 48 for the older working cohort and also 48 for the younger. In the Period Two, the income pattern is changed.[18] From then on, the wage is 72 for the older cohort and 24 for the younger. Thus, the wage sum, i.e. the contribution base, is constantly 96. Also, the average wage for workers in general remains unaltered; only the distribution of the average wage between the age groups have shifted.

The rules of the pension system. The pension system is designed to pay a benefit that is 50 percent of the gross average wage of all wage earners – admittedly an awkward rule, but here it serves our purpose.

The effects of the shift in income pattern. This system will result in contributions of 24 that perfectly match pension benefits of 24 before and after the shift in income pattern. Cohort B, the only cohort whose lifetime income is altered by the change in income pattern, will receive a pension of 24, whereas it paid contributions of 30, the sum of 25 percent of wages 48 and 72, respectively. As the pension received is only 24, this cohort will receive 6 less than they paid, i.e. a periodically compounded rate of return of roughly minus 15 percent.[19]

Table 1A. Effect of a shift in income pattern on cohort contributions and benefits
Central box show wage sums in normal type and *pensions in italic*, per period, for each cohort

Cohort \ Period		0	1	2	3	4	Cohort total: Contributions	Cohort total: *Pensions*
A		48	48	*24*			24	*24*
B			48	72	*24*		30	*24*
C				24	72	*24*	24	*24*
D					24	72	24	..
Period total	Wage sum	..	96	96	96	..		
	Contrib. rate	25 %	25%	25%	25%	25%		
	Contributions	..	24	24	24	..		
	Pensions	..	..	*24*	*24*	*24*		

The effect of the change in income pattern on the system's cross-section rate of return is the monetary effect, - 6, relative to the systems pension liability[20], 36. Thus the cross-section rate of return is minus 1/6. Table 1B. shows that the cross-section rate of return is equal to the relative decrease in the money-weighted average difference in time between the payment of contributions and the collection of benefits, i.e. the decrease in turnover duration from 1.5 to 1.25. From table 1B it is also clear that the change in turnover duration

[18] Income pattern is defined as the ratio of the average wage for each age group to the average wage for all age groups. A stable income pattern exists when this ratio is constant over time for all age groups.

[19] $25\% r^2 + 25\% \frac{3}{2} r = 50\% \Rightarrow r = 0.85078\ldots, \left(-2.35078\ldots \right) \Rightarrow r - 1 \approx -15\%$

[20] Pension liability, PL, is defined in Section 2 Eq. 3 as the present value of future pension benefits to all persons to whom the system has a liability at the time of evaluation, minus the present value of future contributions by the same individuals. As there is neither population growth nor technological progress, the contribution base will be constant; thus, the discount rate will be zero.

makes the contribution asset, i.e. the contribution flow times the turnover duration, decrease and that this decrease is equal to the monetary loss incurred by cohort B.

Table 1B. Effect of a shift in income pattern on turnover duration and pension liability

	Before shift	After shift	Relative change
Av. Age of retiree, $\overline{A}_R$	3[a]	3[a]	-
Av. Age of contributor, $\overline{A}_C$	1.5[b]	1.75[c]	+ 1/6
Turnover duration, TD, ($\overline{A}_R$ - $\overline{A}_C$)	1.5	1.25	- 1/6
Contribution asset, TD x contributions	36	30	-1/6
Pension liability, PL	36[d]	30[e]	- 1/6
IRR (Monetary loss / PL)	- 6/36		- 1/6

[a] All pensions are paid at **age 3**, [b] (48 x **2** + 48 x **1**) / (48+48), [c] (72 x **2** + 24 x **1**) / (72 + 24).
[d] [*24*] + [*24* – 12],[e] [*24*] + [*24* – 18]. In explanations [b], [c], and [d] contributions are shown in normal type, *pensions in italics*, **ages in bold-face**. For explanation of pension liability, brackets [ ] are used to group money flows from and to the same cohort. Figures relating to cohorts are presented in order from oldest to youngest.

The shift in income pattern, together, with the rules of this pension system, makes the pension liability decreases to the same extent as the value of the contribution flow is reduced by the shorter turnover duration. Before the shift, i.e. of period 1, the pension liability was 36 after the shift, i.e. period 2, the pension liability is 30. Owing to this implicit negative indexation of the pension liability, the net present value of the system is consistently zero throughout the shift. By the rules of the system, the negative IRR is implicitly distributed to the “shift” Cohort B. However the negative IRR itself was not a consequence of the system’s rules, as will be illustrated in the following example.

Example 1.1 Same Shift in Income pattern, Different Pension System Rules

The rules of the pension system. The same shift in income pattern is now applied in a pension system designed as a so-called notional defined-contribution (NDC) system. The rules of such a system imply that each cohort is to be repaid an amount equal to their contributions indexed at some rate. Initially, the indexing rules of the system are assumed to provide that notional pension capital and pensions are to be revalued at the growth rate of the contribution base, which in the example is zero for every period.

The effects of the shift in income pattern. Up until and including Cohort A and Period Two this system will yield an identical result as the first set of rules – zero cross-section and longitudinal internal rates of return. But when Cohort B retires, it will have accumulated a notional pension capital of 30, equal to what it has paid in contributions. As the flow of contributions is constant at 24, the system can only repay Cohort B their notional pension capital by incurring a deficit of 6 – a figure familiar from Example 1. This deficit is caused by the same reduction in turnover duration as in Example 1. However, in the notional defined-contribution system the same negative IRR causes a cash deficit since the “rate of implicit revaluation” is zero, while in Example 1 it was minus 1/6, matching the negative IRR.

The shift in income pattern does not immediately reduce the pension liability of the notionally defined-contribution system, it remains at 36 Period Two[21], while the shorter turnover duration – just as in Example 1 – has decreased the value of the contribution flow to 30. To be financially stable, the notional defined-contribution pension system must explicitly distribute the negative IRR by reducing the pension liability. One way to accomplish this is to index the system's total pension liability by the "rate of available indexation", see Eq. 16. Table 1C shows the development of the income statement and balance sheet of the NDC system which applies indexation at the available rate, here equal to the IRR.

Indexing Cohort B's and C's notional pension capital of 30 and 6[22] respectively, by the available rate 5/6 reduces it to 25, and 5, respectively. Thus the total pension liability of the system is reduced from 36 to 30 – equal to the new shorter turnover duration of the system (1.25) times the contribution flow (24). This implies that the system has regained its zero net present value. Nonetheless, the shift in income pattern and the negative indexation of the pension liability will affect the system's cash flows. Period Three pension payments to cohort B will be 25. As system income consistently is 24 this will result in a deficit of 1. In Period Four pension payments to cohort C will be 23 (5 + 18); thus a cash flow surplus of 1 will arise and close the deficit.[23] The systems total assets Period Three are 29, i.e. the sum of the buffer fund Period Three is -1 and the contribution asset is 30. The total assets are equal to the system's pension liability, and the system's net present value is consistently zero.

Table 1C. Effect of a shift in income pattern, income statement and balance sheet
NDC system and indexing at the available rate, in the example equal to the IRR

[21] Pension liability to Cohort B is 30, and to Cohort C it is 6. Only after Cohort B has passed through the system the total pension liability will drop to the new sustainable level of 30 – disregarding the deficit of 6 caused by the shift in income pattern.
[22] The total wage of Cohort C period 2 is ½, with the contribution rate of 25 percent this will make Cohort D's contribution and notional capital equal 0.125.
[23] The return on the buffer fund assets is assumed to equal the growth in contribution base which is zero.

Period	1	*2**	2	3	4
Income statement					
Contributions	24	*24*	24	24	24
Pensions	- 24	*- 24*	- 24	- 25	- 23
= Net cash flow	0	*0*	0	- 1	1
Value of change in turnover duration	0	*- 6*[a]	- 6[a]	0	0
New accrued pension liability♣	- 24	*- 24*	*- 24*	- 24	24
Paid-off pension liability♣ (= paid pension benefits)	24	*24*	*24*	25	23
Indexation of liability♣	0	*0*	*6*	0	0
Net income/ - loss	0	*- 6*	*0*	0	0
Balance sheet					
Buffer fund	0	*0*	0	- 1	0
Contribution asset	36	30	30	30	30
Total assets	36	30	30	29	30
Pension liability, age 3	0	0	0	0	0
Pension liability, age 2	24	30	25[b]	23[c]	24
Pension liability, age 1	12	6	5[c]	6	6
Total liability	23	36	30	29	30
Net present value of system	0	- 6	0	0	0

* Values before indexation with the available rate of return.

♣ A negative number (-) denotes an increase in the pension liability increases and thus a cost. A positive number denotes a decrease in the pension liability decreases and thus income.

[a] - 0.25 [change in TD] x 24 [(contributions(t) + contributions(t-1))/2] = - 6.

[b] (12 [Cohort B's Period One contribution] + 18 [Cohort B's Period Two contribution]) x 5/6 [IRR] = 25.

[c] 6 [Cohort C's Period Two contribution] x 5/6 [IRR] + 18 [Cohort C's Period Three contribution] = 23.

Example 2. A Shift in Mortality Pattern

Summary of what the example illustrates. In example 2 the mortality pattern shifts - life expectancy shifts upwards - so that the average age of retirees increases. It is shown how this increases the turnover duration and causes a positive IRR. The effects of the positive IRR are illustrated for a pensions system with rules that are such that the rate of implicit indexation equals the positive IRR. In example 2.1 the effects of the same shift in mortality pattern is illustrated for a system with rules that are such that the rate of implicit indexation is zero. Thus in example 2.1, to maintain a zero net present value, the positive IRR must be distributed by an *explicit* indexation equal to the IRR. The subsequent effects on the systems cash flows, buffer fund, etc are illustrated by means of an income statement and balance sheet.

The shift in mortality pattern. The shift in mortality occurs – simply though unrealistically – through a one-time increase in life span. After one period of retirement, no retiree in Cohort B dies; instead, after the third period, all retirees in the cohort continue to live for exactly one more period. Subsequent cohorts also live for exactly two periods as retirees.

The rules of the pension system. In the example, the contribution rate remains fixed at 25 percent, and period pension benefits are halved after the first cohort with a longer life expectancy received its first pension payment. Cohort B's pension is thus 24 in their first period as retirees and 12 in their second. Cohort C, the second cohort with a longer life span, will receive a pension of 12 in each period, as will subsequent cohorts.

The effects of the shift in income pattern. Also this system will result in contributions of 24 that perfectly match pension benefits of 24 before and after the shift in mortality. Cohort B, the first to benefit from the longer life span, will receive a total pension of 36, whereas it paid only 24 in contributions, for a periodically compounded rate of return of approximately 25 percent.[24]

Table 2A. Effect of a shift in mortality on cohort benefits
Central box show wage sums in normal type and *pensions in italic*, per period, for each cohort

Cohort \ Period		-1	0	1	2	3	4	Cohort total Contrib.	Cohort total *Pensions*
A		48	48	*24*				24	*24*
B			48	48	*24*	*12*		24	*36*
C				48	48	*12*	*12*	24	*24*
D					48	48	*12*	24	...
Period total	Wage sum	..	96	96	96	..	..		
	Contrib. rate	25 %	25%	25%	25%	25%	25%		
	Contributions	..	24	24	24	..	..		
	Pensions	..	..	*24*	*24*	*24*	*24*		

The effect of the change in mortality pattern on the system's cross-section rate of return is Cohort B's monetary gain, 12, relative to the systems pension liability 36. Thus the cross-section rate of return is 1/3. Table 2B shows that the cross-section rate of return is equal to the relative increase in the money-weighted average difference in time between the payment of contributions and the collection of benefits, i.e. the increase in turnover duration from 1.5 to 2. The reason for the positive return is the longer time span between the average wage-weighted age of contributors and the average benefit-weighted age of retirees resulting from the shift in mortality pattern, i.e. the increase in turnover duration. With the longer turnover duration, the value of the contribution flow increases from 36 to 48.

Table 2B. Effect of a shift in mortality pattern on turnover duration and pension liability

	Before shift	After shift	Relative change
Av. Age of retiree, $\overline{A}_R$	3[a]	3.5[f]	+ 1/6
Av age of contributor, $\overline{A}_C$	1.5[b]	1.5[b]	-
$\overline{A}_R$ - $\overline{A}_C$, turnover duration, TD	1.5	2	+ 1/3
Contribution asset, TD x contributions	36	48	
Pension liability, *PL*	36[d]	48[g]	+ 1/3
IRR (monetary gain / *PL*)	12/36		+ 1/3

[a, b, d] see table 1B. [f] (*12* x **4** + *12* x **3**) / (*12* + *12*), [g] [*12*] + [*12* + *12*] + [*12* + *12* – 12]. See table 1.B for explanation of the use of regular type, *italics* and **bold-face**.

[24] $25\% r^3 + 25\% r^2 = 50\% r + 25\% \Rightarrow r = 1.2469..., (-1.8019..., -0.4450...) \Rightarrow r - 1 \approx 25\%$

The system is financially balanced throughout the shift since the pension liability increases to the same extent as the value of the contribution flow. The positive return of 12 is implicitly distributed to the cohort whose initial pension was calculated on the basis of the previous life expectancy. This can also be illustrated by placing the numbers in the example into Eq. 16;

0 [rate of available indexation] = 0 [i] + 1/3 [ii] + 0 [iii] – 1/3 [rate of implicit revaluation].

The positive return resulting from an increase in life expectancy is due neither to the design of the pension system, nor to the imperfect knowledge of life expectancy assumed in the example. If Cohort B’s life-span had been known *ex ante* and the benefit had been reduced to 12 already in Cohort B’s first period of retirement, there would have been a surplus of 12 in Period Two. Eq. 16 would then read as follows;

1/3 [rate of available indexation] = 0 [i] + 1/3 [ii] + 0 [iii] – 0 [rate of implicit revaluation].

If the available indexation is not used to increase the pension liability the identity requirement for financial stability – a zero net present value – is not met since an undistributed surplus then arises. Example 2.1 illustrates the effects of one rule to distribute this surplus.

Example 2.1 Same Shift in Mortality Pattern, Different Pension System Rules

The rules of the pension system. We now again assume a NDC system which indexes its liability by the available rate of return. In the example, this return will equal the IRR since we assume perfect information on life expectancy. In such a system and with this information, the surplus of 12, representing a rate of available indexation of 1/3, will be distributed through indexation of the pension liability Period Two.

The effects of the shift in mortality pattern. Pension payments will be 16, 30, 26 in Periods, Three, Four and Five, respectively. Pension payments will be stable at 24 as from Period Six. Periods Three and Four, the buffer fund will thus stand at 8 and 2, respectively and be back at 0 as from Period Five. (Readers are encouraged to verify these calculations). The positive fund is necessary to balance the pension liability, which will temporarily exceed the contribution asset by the same magnitude as the value of the fund. Assuming revaluation of the pension liability at the rate of available indexation and, more realistically, imperfect information on life expectancy, the flow of payments will be different. Still, the system will maintain a zero net present value at all times and in a steady state end up with a zero buffer fund.

Table 2.C Effect of a shift in mortality, income statement and balance sheet
NDC system and indexing at the available rate, in the example equal to the IRR

Period	1	*2**	2	3	4	5
Income statement						
Contributions	24	*24*	24	24	24	24
Pensions	- 24	*- 24*	- 24	- 16[e]	- 30[f]	- 26[g]
= Net cash flow	0	*0*	0	8	- 6	- 2
Value of change in turnover duration	0	*12*[a]	12[a]	0	0	0
New accrued pension liability♣	- 24	*- 24*	*- 24*	- 24	- 24	- 24
Paid-off pension liability♣ (= paid pension benefits)	24	*24*	*24*	- 16		2
Cost of / income from indexation of liability♣	0	*0*	*-12*	0	0	0
Net income/ - loss	0	*12*	*0*	0	0	0
Balance sheet						
Buffer fund	0	*0*	*0*	8	2	0
Contribution asset	36	*48*	48	48	48	48
Total assets	36	*48*	48	56	50	48
Pension liability, age 3	0	*0*	0	16	14	12
Pension liability, age 2	24	*24*	32[b]	28[d]	24	24
Pension liability, age 1	12	*12*	16[c]	12	12	12
Total liability	36	*36*	48	56	50	48
Net present value of system	0	12	0	0	0	0

*, ♣ See Table 1C.

[a] 0.5 [change in TD] x 24 [(contributions(t) + contributions(t-1))/2] = 12.

[b] (12 [Cohort B's Period One contribution] + 12 [Cohort B's Period Two contribution]) x 4/3 [IRR] = 32.

[c] 12 [Cohort C's Period Two contribution] x 4/3 [IRR] = 16.

[d] 12 [Cohort C's Period Two contribution] x 4/3 [IRR] = 16 + 12 [Cohort C's Period Three contribution] = 28.

[e] Cohort B's pension Period Three; 32 [notional pension capital] / 2 [life expectancy] = 16.

[f] 16 [Cohort B's pension Period Four] + ; Cohort C's pension Period Four; 28 [notional pension capital] / 2 [life expectancy] = 30.

[g] 14 [Cohort C's pension Period Five] + ; Cohort D's pension Period Five; 24 [notional pension capital] / 2 [life expectancy] = 26.

APPENDIX B. INTERMEDIATE STEPS BEFORE EQUATION 7

$$\frac{V}{C}=\frac{N(0)\cdot\overline{W}\cdot k\cdot\int_0^m\int_x^m l(u)\cdot e^{-\gamma\cdot u}\cdot\left[R(u)\cdot e^{\varphi\cdot u}-W(u)\cdot\frac{\int_0^m e^{-(\gamma-\varphi)\cdot a}\cdot l(a)\cdot R(a)\,da}{\int_0^m e^{-\gamma\cdot a}\cdot l(a)\cdot W(a)\,da}\right]du\,dx}{N(0)\cdot\overline{W}\cdot k\cdot\int_0^m l(x)\cdot e^{-\gamma\cdot x}\cdot W(x)\cdot\frac{\int_0^m e^{-(\gamma-\varphi)\cdot a}\cdot l(a)\cdot R(a)\,da}{\int_0^m e^{-\gamma\cdot a}\cdot l(a)\cdot W(a)\,da}dx}\ldots$$

The expression above can easily be reduced by elementary algebraic manipulations. However, to make it simpler the following substitutions will be useful:

$$\begin{Bmatrix}F_R(a)=e^{-(\gamma-\varphi)\cdot a}\cdot l(a)\cdot R(a)\\ F_W(a)=e^{-\gamma\cdot a}\cdot l(a)\cdot W(a)\end{Bmatrix}$$

$$\ldots=\frac{\int_0^m\int_x^m l(u)\cdot e^{-\gamma\cdot u}\cdot\left[R(u)\cdot e^{\varphi\cdot u}-W(u)\cdot\frac{\int_0^m F_R(a)\,da}{\int_0^m F_W(a)\,da}\right]du\,dx}{\int_0^m F_W(x)\cdot\frac{\int_0^m F_R(a)\,da}{\int_0^m F_W(a)\,da}dx}$$

$$=\frac{\int_0^m\int_x^m l(u)\cdot e^{-\gamma\cdot u}\cdot\left[R(u)\cdot e^{\varphi\cdot u}\cdot\int_0^m F_W(a)\,da-W(u)\cdot\int_0^m F_R(a)\,da\right]du\,dx}{\int_0^m F_R(a)\,da\cdot\int_0^m F_W(x)\,dx}$$

$$=\frac{\int_0^m\left[\int_0^m F_W(a)\,da\cdot\int_x^m l(u)\cdot e^{-\gamma\cdot u}\cdot R(u)\cdot e^{\varphi\cdot u}\,du-\int_0^m F_R(a)\,da\cdot\int_x^m l(u)\cdot e^{-\gamma\cdot u}\cdot W(u)\,du\right]dx}{\int_0^m F_R(a)\,da\cdot\int_0^m F_W(x)\,dx}$$

$$=\frac{\int_0^m\left[\int_0^m F_W(a)\,da\cdot\int_x^m F_R(u)\,du-\int_0^m F_R(a)\,da\cdot\int_x^m F_W(u)\,du\right]dx}{\int_0^m F_R(a)\,da\cdot\int_0^m F_W(x)\,dx}$$

$$=\frac{\int_0^m F_W(a)\,da\cdot\int_0^m\left[\int_x^m F_R(u)\,du\right]dx-\int_0^m F_R(a)\,da\cdot\int_0^m\left[\int_x^m F_W(u)\,du\right]dx}{\int_0^m F_R(a)\,da\cdot\int_0^m F_W(x)\,dx}$$

$$=\frac{\int_0^m\left[\int_x^m F_R(u)\,du\right]dx}{\int_0^m F_R(x)\,dx}-\frac{\int_0^m\left[\int_x^m F_W(u)\,du\right]dx}{\int_0^m F_W(x)\,dx}\ldots$$

For the next step we need the following identity:

$$\left\{\begin{array}{l}\int_0^m \int_x^m f(u)\,du\,dx = \int_0^m x\cdot f(x)\,dx \\ \text{proof:} \\ \int_0^m \int_x^m f(u)\,du\,dx = \left[x\cdot\int_x^m f(u)\,du \right]_0^m - \int_0^m x\cdot\left(-f(x)\right)dx \\ = m\cdot\int_m^m f(u)\,du - 0\cdot\int_0^m f(u)\,du + \int_0^m x\cdot f(x)\,dx = 0+0+\int_0^m x\cdot f(x)\,dx \end{array}\right\}$$

thus,

$$\ldots = \frac{V}{C} = \frac{\int_0^m x\cdot F_R(x)\,dx}{\int_0^m F_R(x)\,dx} - \frac{\int_0^m x\cdot F_W(x)\,dx}{\int_0^m F_W(x)\,dx} = \frac{\int_0^m x\cdot e^{-(\gamma-\varphi)\cdot x}\cdot l(x)\cdot R(x)\,dx}{\int_0^m e^{-(\gamma-\varphi)\cdot x}\cdot l(x)\cdot R(x)\,dx} - \frac{\int_0^m x\cdot e^{-\gamma\cdot x}\cdot l(x)\cdot W(x)\,dx}{\int_0^m e^{-\gamma\cdot x}\cdot l(x)\cdot W(x)\,dx}$$

QED

APPENDIX C. ROUGH ESTIMATE OF EXPECTED TURNOVER DURATION IN 41 COUNTRIES

Individuals that are not in the labor force and are 55 years or older are assumed to receive benefits from the pension system that on average amount to 50 % of the average wage; pensions are assumed to be indexed by the growth in the average wage, i.e. $\varphi = 0$.

Country	Year	Est. pop. growth, $\gamma-1$, per cent	Expected turnover duration, TD, years
Tajikistan	1991	3.8	35.3
Argentina	1990	1.5	34.1
Spain	1990	0.2	34.1
New Zealand	1990	0.8	34.0
Australia	1994	0.3	34.0
Kyrgyzstan	1995	2.8	33.7
Israel	1994	1.8	33.6
Portugal	1992	0.1	33.3
Canada	1992	-0.1	33.3
Chile	1997	1.5	33.2
Romania	1992	0.5	33.2
Italy	1994	-0.3	33.2
U.S.	1995	0.2	33.1
Austria	1996	-0.4	33.1
Belgium	1994	-0.1	33.1
Ireland	1990	0.9	33.0
France	1995	0.2	33.0
U.K.	1996	-0.1	32.9
Hungary	1996	-0.1	32.8
Greece	1995	-0.1	32.8
Kazakhstan	1996	1.4	32.7
Slovakia	1995	0.8	32.7
Denmark	1994	-0.4	32.7
Sweden	1996	-0.3	32.7
Netherlands	1995	-0.3	32.7
Latvia	1996	-0.1	32.6
Norway	1996	0.1	32.6
Armenia	1993	1.6	32.5
Czech Rep.	1996	-0.1	32.1
Slovenia	1993	-0.2	32.0
Estonia	1996	0.0	31.9
Belarus	1996	0.1	31.9
Poland	1996	0.4	31.8
Russian Fed.	1995	-0.1	31.7
Germany	1994	-0.8	31.7
Bulgaria	1993	-0.2	31.6
Japan	1990	-0.4	31.6
Finland	1996	-0.4	31.6
Korea Rep.	1991	0.9	31.5
Ukraine	1993	-0.1	31.3
Moldova	1994	0.8	31.2

Source: adapted UN and ILO statistics. For details see Settergren & Mikula (2001).

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