Choice, Opportunities, and Procedures: Collected Papers of Kotaro Suzumura

Part III Social Choice and Welfare Economics

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January 4, 2007

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Chapter 9
Impossibility Theorems without Collective Rationality*

1 Introduction

Arrow's general impossibility theorem [2] demonstrated the incompatibility of five conditions on collective choice rules: *unrestricted domain, nondictatorship, the Pareto condition, independence of irrelevant alternatives*, and *transitive rationality* of social choice function. The last condition requires the existence of a social preference ordering such that, given a set of alternatives, the chosen elements are those which are best with respect to that ordering. Since the publication of Arrow's theorem, an extensive body of literature has appeared seeking to circumvent the difficulty. This chapter focuses on attempts to resolve the paradox by weakening the collective rationality requirement.

For our purpose, it is convenient to decompose Arrow's collective rationality requirement into two parts:

(a) Rationality. There exists a social preference relation R such that the elements chosen out of a set of available alternatives S are those which are best in S with respect to R. (R will be referred to as a *rationalization*.)

(b) Transitivity and completeness of the rationalization.

The sensitivity of Arrow's result to the specification of the degree of rationality was first noticed by Sen [22]. He continued to impose rationality but relaxed the second component to require only completeness and quasi-transitivity (that is, transitivity of strict preference), and showed that this weakened collective rationality requirement is compatible with the remainder of Arrow's conditions. Gibbard [10] subsequently proved that any society whose collective choice rule meets Sen's conditions contains an *oligarchy*, a

*First published in *Journal of Economic Theory*, Vol.13, 1976, pp.361-379. Joint paper with D. H. Blair, G. Bordes and J. S. Kelly. Reprinted in Arrow, K. J. and G. Debreu, eds., *The Foundations of 20th Century Economics, Vol.3, Landmark Papers in General Equilibrium Theory, Social Choice and Welfare*, Cheltenham, Glos: Edward Elgar, 2001, pp.660-678. This paper represents a consolidation of overlapping work done independently by the four authors [3, 5, 13, 26]. The authors are indebted to the referees of the *Journal of Economic Theory, the Review of Economic Studies,* and *Econometrica* for seeing the possibility of such a consolidation. Thanks are also due to Donald J. Brown, John A. Ferejohn, Robert P. Parks, and Amartya K. Sen.

class of individuals who are jointly decisive for exclusion of an alternative from the social choice out of a two-element set and each of whose members is individually decisive for inclusion of an element in the choice from such a set. Any individual who by strictly preferring x to y can ensure that y is not socially preferred to x is called a *weak dictator*; every member of an oligarchy is clearly a weak dictator. Mas-Colell and Sonnenschein [14] provided the first published proof of Gibbard's theorem and proved an alternative impossibility result: even if weak dictators are to be countenanced, their multiplicity causes quasi-transitive rational (and otherwise Arrovian) collective choice rules to violate a decisiveness condition they call *positive responsiveness*. Thus, demanding merely quasi-transitive rationality of social choice provides no satisfactory resolution of Arrow's antidemocratic result. Even the smallest nondictatorial oligarchy (of two) fails a requirement of responsiveness (which is admittedly quite strong) when there are more than two voters; enlarging and "democratizing" the oligarchy aggregates the heterogeneity of individual preferences into widespread social indifference rather than intransitivity.

Further weakening of the consistency requirement imposed by Arrow's collective rationality (while continuing to insist on the existence of a rationalization) is entailed by requiring acyclicity (nonexistence of a strict preference cycle) instead of quasi-transitivity of the social preference relation. The importance of this substitution comes from the observation that acyclicity is necessary and sufficient to guarantee that society is able to make a nonempty rational choice from any *finite* subset of the set of alternatives. In the case of individuals with acyclic preferences choosing over an *infinite* set of alternatives, Brown [7] has shown that the only acyclic collective choice rules which satisfy the remainder of Arrow's conditions and are not oligarchic are those of what he calls *collegial polities*. Under such a procedure, there exists a *quasi-oligarchy*, a subset of individuals whose unanimous assent is a necessary condition for the exclusion of an alternative from the social choice out of a two-element set. In contrast with the Gibbardian oligarchy, consensus within the quasi-oligarchy, though necessary, is not sufficient for exclusion. For this class of decision rules, at least one individual outside the quasi-oligarchy must also prefer x to y to ensure a similar social preference. Thus, weakening Arrow's transitive rationality to require only acyclic rationality is a step in the democratic direction. The complete asymmetry between the power of individuals within and outside the oligarchy is diluted when quasi-transitivity is abandoned. Some non-quasi-oligarchs do have power: they are pivotal to the success of some winning coalitions. Nevertheless the tradeoffs remain between heterogeneity of preferences, decisiveness, and inequalities in the distribution of power, as is shown by another Mas-Colell-Sonnenschein theorem which asserts that no acyclic collective choice rules exist satisfying *both* their no-weak-dictators and positive responsiveness conditions along with the remainder of Arrow's conditions. This proposition imposes no restrictions on the size of the alternatives set. In the case of individuals with acyclic preferences choosing over a finite set, Brown [6] has obtained a precise characterization of acyclic Arrovian collective choice rules which indicates clearly how they violate the positive responsiveness requirement. Collegial polities are of course acyclic even on a finite set, and nontrivial ones are obviously unresponsive to changes in the preferences of some voters. The only anonymous acyclic procedures in the finite case, as Brown shows, are rules satisfying the following condition: if M is the number of alternatives, every M-tuple of decisive coalitions of individuals must have nonempty intersection. This class of procedures includes simple special-majority rules (e.g., $\frac{2}{3}$ majority) and representative systems with a special-majority rule at each stage. For alternative sets which are large relative to the size of the set of individuals, these procedures are close to the unanimity rule.

Given all these results involving the weakening of Arrow's transitivity requirement, it is not surprising to find attacks focusing directly on rationality itself. Both Schwartz [21] and Plott [16; 17; 18] have criticized the demand for the existence of rationalizations. Plott [18] argues that a major reason for Arrow's insistence on transitive rationality was that it ensured that the social choice would be invariant under arbitrary manipulations of the agenda, that is, the order and method by which alternatives are compared and inferior ones discarded (see Arrow [2, p.120]). He proposes a consistency requirement for choice functions which he calls *path independence*:

> The alternatives are "split up" into smaller sets, a choice is made over each of these sets, the chosen elements are collected, and then a choice is made from them. Path independence, in this case, would mean that the final result would be independent of the way the alternatives were initially divided up for consideration (Plott [17, pp.1079-1080]).[1]

A fairly natural question now arises: What happens to impossibility theorems when path independence is substituted for transitive rationality and the remaining Arrovian conditions prevail? Plott [17, 18] observes, citing Sen [22] as a source, that the collective choice rule which chooses the Pareto optimal subsets from available alternatives sets serves as a counterexample to a proposed impossibility result. Unfortunately this collective choice rule runs afoul of Gibberd's theorem; it is also too undiscriminating in the face of heterogeneous individual preferences. It is important to notice that there exist path-independent choice functions which have no rationalization. Plott's position on impossibility results *with* path independence but *without* rationality is ambiguous. He has said that "some of the standard constructions in welfare economics such as social welfare functions and social preference relations unduly restrict the set of admissible policies and consequently induce impossibility results" (Plott [16, p.182]) and that, with the relaxation of rationality, "the immediate impossibility result discovered by Arrow is avoided" (Plott [17, p.1075]). He has been careful, however, to observe that "the lines which separate rationality properties, which induce immediate impossibility results, from path independence properties are very thinly drawn" [17, p.1075]. Blair [4] and Parks [15] have exhibited examples of collective choice rules which can result in path-independent but not quasi-transitive choices; both, however, suffer from the defect that they can generate choice functions which are not very selective.

[1] For procedures aggregating preferences over many alternatives, which must of necessity be multistage processes due to computational costs, path independence is a desirable property for two reasons. First, it rules out certain forms of institutional arbitrariness, such as bias in favor of the status quo. Second, it precludes strategic behavior at the agenda-determination stage.

We prove in this paper several impossibility theorems in which we do not require social choices to have a rationalization. One of our results shows the incompatibility of non-weak-dictatorship, the Pareto condition, independence of irrelevant alternatives, and path independence. Thus the replacement of Arrow's collective rationality with Plott's path independence does not help us to escape from the Arrovian dilemma. Still weaker consistency properties for social choice functions than path independence will be proposed. They too, however, fail to provide us with a means of avoiding impossibility results. As Arrow has written, "the paradox of social choice cannot be so easily exorcised" [2, p.109].

2 The Structures of Choice Functions

Before presenting our impossibility theorems, we will clarify in this section the relationships between, on one hand, the rationality conditions used in the existing impossibility theorems and, on the other, path independence and some weaker conditions.

Let X denote the set of (mutually exclusive) alternatives. K stands for a family of nonempty subsets of X. Each element $S \in K$ is an *admissible agenda*; it contains the currently feasible alternatives in a given choice situation. We assume throughout this chapter that K contains all nonempty finite elements of $\mathcal{P}(X)$, the power set of X. A *choice function* C on K is a function which maps each $S \in K$ into a nonempty subset $C(S)$ of S; note that $C(S)$ is not required to be a one-element set: Five properties of choice functions will be of interest here:

Path independence (PI). $C(S_1 \cup S_2) = C(C(S_1) \cup C(S_2))$ for all $S_1, S_2 \in K$.

Chernoff condition (C). $S_1 \subset S_2 \Rightarrow C(S_2) \cap S_1 \subset C(S_1)$ for all $S_1, S_2 \in K$. That is, every element chosen out of a set must also be chosen in every subset of the set containing the element.[2]

Property β. $[S_1 \subset S_2 \;\&\; C(S_1) \cap C(S_2) \neq \emptyset] \Rightarrow C(S_1) \subset C(S_2)$ for all $S_1, S_2 \in K$. That is, if some chosen element from a set is chosen from a superset of that set, then *every* such element is chosen from the superset. (See Sen [24].)

Superset property (S). $S_1 \subset S_2 \Rightarrow$ not $[C(S_2) \subsetneq C(S_1)]$. That is, the choice out of the superset of a set is not strictly contained in the choice out of the set.

Generalized Condorcet property (GC). ($x \in S$ & $x \in C(\{x, y\})$ for all $y \in S$) $\Rightarrow x \in C(S)$ for all $S \in K$. That is, if no element in a set beats a given element x in a binary choice, then x must be among the elements chosen from the set.[3]

[2] This condition, first introduced by Chernoff [8], has appeared in the literature under a variety of names including, unhappily, "independence of irrelevant alternatives." It is discussed extensively by Arrow in [1].

[3] The Condorcet condition in its usual form is stated in terms of pairwise comparisons by simple majority rule. Our condition is a weaker version of Sen's property γ, discussed in [24].

A *preference relation* R is a binary relation on X having the interpretation that xRy iff x is at least as good as y from the point of view of the person or group in question. In the usual way we define from R the subrelations P of strict preference and I of indifference. R is *complete* iff xRy of yRx, *transitive* iff xRy & $yRz \Rightarrow xRz$, *quasi-transitive* iff P is transitive, and *acyclic* iff $(x_1Px_2P\dots Px_tPx_1)$ for no finite subset $\{x_1,\dots,x_t\}$ of X. A transitive and complete relation will be called a *transitive* ordering.

Choice functions induce preference relations in two ways. If there exists a preference relation R satisfying $C(S) = \{x \in S : xRy \text{ for all } y \in S\}$ for all $S \in K$, the choice function C will be called *rational* (R); in that event R is a *rationalization* of C. A choice function is *transitive rational* (TR), *quasi-transitive rational* (QTR), or *acyclic rational* (AR) if it has a rationalization with the requisite property.

Alternatively, a preference relation may be derived from choice functions restricted to two-element agenda sets. Even if C has no rationalization, we can always define, following Herzberger [12], the *base relation* R^* as follows: xR^*y iff $x \in C(\{x,y\})$ for all $\{x,y\} \in K$. Strict preference P^* may be defined in the obvious way. A choice function will then be said to satisfy *base quasi-transitivity* (BQT) iff R^* is quasi-transitive, *base acyclicity* (BA) iff R^* is acyclic, and *base triple acyclicity* (BTA) iff xP^*y & $yP^*z \Rightarrow xR^*z$.

We turn now to comparing these consistency conditions by decomposing several of them into more basic parts. Sen [24] has proven that a choice function has a transitive rationalization iff it satisfies both property β and the Chernoff condition. Plott [17], in turn, has shown that quasi-transitive rationality is equivalent to the conjunction of path independence and the generalized Condorcet property. The relationship between these results becomes more apparent when we further decompose path independence.

Theorem 1. *A choice function is path independent if and only if it satisfies the Chernoff condition and the superset property.*

Proof. First we show that path independence implies the Chernoff condition. Let $S_1 \subset S_2$, and let $x \in S_1 \cap C(S_2)$. By path independence, $C(S_2) = C[C(S_2 \setminus S_1) \cup C(S_1)] \subset C(S_2 \setminus S_1) \cup C(S_1)$. Since $x \in S_1, x \notin S_2 \setminus S_1$, so $x \notin C(S_2 \setminus S_1)$. Hence $x \in C(S_1)$; therefore $S_1 \cap C(S_2) \subset C(S_1)$.

Next we show that path independence implies the superset property. Suppose, contrary to that condition, that $S_1 \subset S_2$ and $C(S_2) \subsetneq C(S_1)$. By path independence, $C(S_2) = C[C(S_2) \cup C(S_1)] = C[C(S_1)]$. By the first part of this proof, the Chernoff condition holds, so that from $C(S_1) \subset S_1$ we can derive $C(S_1) \cap C(S_1) \subset C[C(S_1)]$; thus $C[C(S_1)] = C(S_1)$. This yields $C(S_1) = C(S_2)$, a contradiction.

Finally, we obtain path independence from the superset property and the Chernoff condition. Suppose $x \in C(S_1 \cup S_2)$. If $x \in S_1$, the Chernoff condition implies $x \in C(S_1)$; if $x \in S_2$, then $x \in C(S_2)$. Hence $x \in C(S_1) \cup C(S_2) \subset S_1 \cup S_2$. By another application of the Chernoff condition, $x \in C[C(S_1) \cup C(S_2)]$. Thus $C(S_1 \cup S_2) \subset C[C(S_1) \cup C(S_2)]$. The inclusion cannot be strict, however, because of the superset property and the fact that $C(S_1 \cup S_2) \subset C[C(S_1) \cup C(S_2)]$. Therefore, $C(S_1 \cup S_2) = C[C(S_1) \cup C(S_2)]$.

What we now know is that quasi-transitive rationality is equivalent to the conjunction of the Chernoff condition, the superset property, and the generalized Condorcet property, and that if the generalized Condorcet property is no longer required we have path independence. Suppose we retain the Chernoff condition and the generalized Condorcet property but do not require that the superset property hold. Theorem 2 demonstrates that what remains is acyclicity.

Theorem 2. *If a choice function C on K satisfies the Chernoff condition and the generalized Condorcet property and if K contains all finite nonempty subsets of X, it has a unique, complete, reflexive, acyclic rationalization. If a choice function is induced by an acyclic relation, it satisfies the Chernoff condition and the generalized Condorcet property.*

Proof. Beginning with the first proposition, we assume C satisfies the conditions stated. Each two-element subset of X belongs to K, so the only possible rationalization is the base relation R^*:

$$xR^*y \qquad \text{iff} \quad x \in C(\{x,y\}).$$

If $x \in C(S)$, then, by the Chernoff condition, $x \in C(\{x,y\})$ for each $y \in S$. On the other hand, if $x \in C(\{x,y\})$ for all $y \in S$, then $x \in C(S)$ by the generalized Condorcet property. Thus,

$$\begin{aligned} C(S) &= \{x \in S : x \in C(\{x,y\}) \text{ for all } y \in S\} \\ &= \{x \in S : xR^*y \text{ for all } y \in S\}, \end{aligned}$$

i.e., R^* is in fact a rationalization of C. Completeness and reflexivity of R^* are obvious: it remains to show that R^* is acyclic. Suppose that $x_1P^*x_2P^* \ldots P^*x_n$, i.e., $x_i \notin C(\{x_{i-1}, x_i\})$ for $i = 2, \ldots, n$. By the Chernoff condition, $x_i \notin C(\{x_1, x_2, \ldots, x_n\})$ for $i = 2, \ldots, n$. By our assumption about the content of K, $C(\{x_1, x_2, \ldots, x_n\}) \neq \emptyset$, so $C(\{x_1, x_2, \ldots, x_n\}) = \{x_1\}$. By another application of the Chernoff condition, $x_1 \in C(\{x_1, x_n\})$, that is, *not* $x_nP^*x_1$, as was to be shown.

Turning now to the second assertion, for each $S \in K$,

$$C(S) = \{x \in S : xRy \text{ for all } y \in S\} = \{x \in S : \text{not } (\exists y)(y \in S \;\&\; yPx)\},$$

where R is acyclic. Suppose $x \in C(\{x,y\})$ for all $y \in S$. Then xRy for all $y \in S$, that is, $x \in C(S)$; the generalized Condorcet property therefore holds. Suppose $x \in C(S_2)$ and $x \in S_1 \subset S_2$. Now $x \in C(S_2)$ implies xRy for all $y \in S_2$, which implies xRy for all $y \in S_1$. Hence $x \in C(S_1)$, and the Chernoff condition holds.

Theorems 1 and 2, coupled with Plott's theorem, imply that the only path-independent choice functions which are acyclic rational are those which are quasi-transitive rational as well. We have earlier remarked that path-independent choice functions exist which are not rational (see Plott [17]). It should now be clear that there exist rational choice functions which violate path independence; indeed, the choice function induced by any acyclic but not quasi-transitive preference relation falls in this category.

The following example shows that the Chernoff condition implies neither path independence nor the existence of a rationalization:

Example. Let $X = \{x, y, z\}$ and $K = \mathcal{P}(X) \setminus \{\emptyset\}$. The choice function defined by $C(X) = \{x\}$ and $C(S) = S$ for all $S \subsetneq X$ is easily shown to satisfy the Chernoff condition. It is not path-independent, however, since $C(\{x, y, z\}) = \{x\} \subsetneq \{x, y\} = C(\{x, y\})$, which contradicts the superset property. If C has any rationalization it must be universal indifference, given $C(S) = S$ for all two-element S, but this contradicts $C(X) = \{x\}$.

Finally, we relate path independence and the Chernoff condition to the properties of the base relation.

Lemma 1. *Path independence implies base quasi-transitivity. The Chernoff condition implies base acyclicity.*

Proof. Suppose that C is path-independent and that xP^*yP^*z for some $x, y, z \in X$. By path independence, $\{x\} = C(\{x, y\}) = C[C(\{x\}) \cup C(\{y, z\})] = C(\{x, y, z\}) = C[C(\{x, y\}) \cup C(\{z\})] = C(\{x, z\})$. Hence xP^*z, so R^* is quasi-transitive.

The second proposition is established by an argument already given in the first part of the proof of Theorem 2.

Counterexamples to the converse of Lemma 1's assertions are left for the reader to construct.

We conclude this section with an implication diagram which summarizes the results presented here and other relationships which follow easily from the definitions. The properties in parentheses are jointly equivalent to the conditions above them. Note that the equivalence of rationality and acyclic rationality is dependent on our assumption that K includes all finite nonempty subsets of X.

$$\begin{tikzcd}
\begin{array}{c}\text{TR}\\(C,\beta)\end{array} \arrow[r] & \begin{array}{c}\text{QTR}\\(C,S,GC)\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{AR}\\(C,GC)\end{array} \arrow[r, leftrightarrow] & \text{R} \arrow[d] & \\
& \begin{array}{c}\text{PI}\\(C,S)\end{array} \arrow[rr] \arrow[d] & & \text{C} \arrow[d] & \\
& \text{BQT} \arrow[rr] & & \text{BA} \arrow[r] & \text{BTA}
\end{tikzcd}$$

3 Impossibility Theorems

Suppose that there are n individuals and let $N = \{1, 2, \ldots, n\}$; it is assumed that $2 \leq n < \infty$. X stands for the set of social alternatives, now taken to have at least three elements. The problem at hand is to characterize the institutionally and ethically "acceptable" collective choice rules; such a rule F is a function which maps each *profile*, or n-tuple of individual transitive preference orderings of X, into the set of choice functions on K. (Note that what is called here a collective choice rule is analogous to Arrow's [2] *social welfare function*, and that choice functions play in this analysis the same role as Arrow's *social preference ordering*.) The domain of F is the set of all logically possible profiles.

Formally, given a profile k, society's choice function C is given by $C = F(k)$. However, the function F will be fixed throughout the proof of each of the subsequent theorems. We will therefore frequently not refer explicitly to F, but rather will simply write $C^k(S)$ for the choice out of agenda set S under profile k, given the fixed collective choice rule. When the profile is invariant, we will suppress the superscript as well and merely write $C(S)$.

All of the rationality and consistency requirements studied in Section 2 are properties of choice functions, that is, of elements of the range of collective choice rules. Each of these conditions will also be attributed to a collective choice rule F in the event that, for every profile, the choice function determined by F satisfies the given condition. For example, we will call a collective choice rule path-independent if the image of every profile under the rule is a path-independent choice function.

A further set of definitions is necessary before proceeding to the results of this section.

A set of individuals $J \subset N$ is *decisive for x against y* (resp. *weakly decisive for x against y*) iff xP_iy for all $i \in J$ and jP_ix for all $i \notin J$ implies $\{x\} = C(\{x, y\})$ (resp. $x \in C(\{x, y\})$).

If V is decisive for some a against some b, and W being decisive for some x against some y implies that the number of individuals in W is at least as great as the number of individuals in V, then V is a *smallest decisive set*.

Individual i is a *dictator (*resp. *weak dictator)* iff for all $x, y \in X, xP_iy$ implies $\{x\} = C(\{x, y\})$ (resp. $x \in C(\{x, y\})$).

A collective choice rule is said to satisfy:

The Pareto condition iff for any profile k such that xP_iy for all $i \in N$, we have $\{x\} = C^k(\{x, y\})$.

Nondictatorship iff there exists no dictator.

Non-weak-dictatorship iff there exists no weak dictator.

Positive responsiveness iff k is a profile resulting in $x \in C^k(\{x, y\})$ and l is another profile with $R^k_j = R^l_j$ for all $j \neq i$ and either $(yP^k_ix$ & $xI^l_iy)$ or $(yI^k_ix$ & $xP^l_iy)$, implies $\{x\} = C^l(\{x, y\})$, where $i \in N$ is any specified individual.

Independence of irrelevant alternatives iff for any two profiles $j = (R_1, \ldots, R_n)$ and $k = (R'_1, \ldots, R'_n)$ such that $(xR_iy \Leftrightarrow xR'_iy$ & $yR_ix \Leftrightarrow yR'_ix)$ for all $i \in N$, we have $C^j(\{x, y\}) = C^k(\{x, y\})$.

Notice that our independence condition restricts its attention to choices from two-element sets, in contrast with Arrow's independence axiom. Although the two conditions are equivalent for rational collective choice rules, our axiom is strictly weaker if rationality is not imposed.

Profiles will be written horizontally with more preferred alternatives to the left; indifference will be indicated by parentheses. For example, the expression

$$V: \qquad x,\ (y, z)$$

means that every individual in V prefers x to both y and z, between which indifference prevails.

We now proceed to establish an impossibility theorem using only path independence rather than Arrow's transitive rationality. The theorem is otherwise Arrovian except for a slight strengthening of the nondictatorship condition.

Theorem 3. *If there are at least three voters, there is no collective choice rule satisfying all of:*

(1) *path independence,*
(2) *the Pareto condition,*
(3) *independence of irrelevant alternatives, and*
(4) *non-weak-dictatorship.*

In view of Lemma 1, we can establish this proposition by proving the following stronger result, which utilizes the weaker condition of base quasi-transitivity. An alternative proof of Theorem 4 is given by Fishburn [9, Theorem 16.2].

Theorem 4. *If there are at least three voters, there is no collective choice rule satisfying all of:*

(1) *base quasi-transitivitiy,*
(2) *the Pareto condition,*
(3) *independence of irrelevant alternatives, and*
(4) *non-weak-dictatorship.*

This proposition follows immediately from the three lemmas below.

Lemma 2. *If a collective choice rule satisfies the Pareto condition, independence of irrelevant alternatives, and base quasi-transitivity, and if i is decisive for some $x, y \in X$, then i is a dictator; that is,*

$$\begin{aligned}&(xP_iy \ \&\ yP_jx \textit{ for all } j \neq i \Rightarrow \{x\} = C(\{x,y\}))\\ &\qquad \Rightarrow (\textit{For all } s,t \in X : sP_it \Rightarrow \{s\} = C(\{s,t\})).\end{aligned}$$

Proof. We first show

$$\begin{aligned}&(xP_iy \ \&\ yP_jx \textit{ for all } j \neq i \Rightarrow \{x\} = C(\{x,y\}))\\ &\qquad \Rightarrow (\textit{For all } s \in X : sP_iy \Rightarrow \{s\} = C(\{s,y\})).\end{aligned} \tag{1}$$

Suppose not. Then there exists a profile such that xP_iy and yP_jx for all $j \neq i$ implies $\{x\} = C(\{x,y\}), sP_iy$, and $\{s\} \neq C(\{s,y\})$:

$i:$ $s, x, y,$
$N \setminus \{i\}$: (some $n-1$-tuple of orderings of y and s), x.

By assumption, $\{x\} = C(\{x, y\})$. By the Pareto condition $\{s\} = C(\{s, x\})$.

Base quasi-transitivity then implies that, under this profile $\{s\} = C(\{s, y\})$. By independence, $\{s\} = C(\{s, y\})$ under *every* profile in which sP_iy, since no specification has been made of other voters' preferences between these alternatives. This contradiction establishes (1). Next we show

$$\begin{aligned}&(\textit{For all } s \in X : sP_iy \;\&\; yP_js \textit{ for all } j \neq i \Rightarrow \{s\} = C(\{s, y\}))\\ &\quad \Rightarrow (\textit{For all } s, t \in X : sP_it \Rightarrow \{s\} = C(\{s, t\})).\end{aligned} \tag{2}$$

The antecedent of (2) is implied by (1). Suppose (2) is false. Then there exists a profile such that sP_iy and yP_js for all $j \neq i \Rightarrow \{s\} = C(\{s, y\}), sP_it$, and $\{s\} \neq C(\{s, t\})$:

$i :$ $s, y, t,$
$N \setminus \{i\}$: y, (some $n - 1$-tuple of orderings of s and t).

By assumption, $\{s\} = C(\{s, y\})$ and by the Pareto condition, $\{y\} = C(\{y, t\})$. Base quasi-transitivity shows that under this profile $\{s\} = C(\{s, t\})$. By independence, the social choice is the same for all profiles in which sP_it holds, contradicting our assumption and establishing (2).

Lemma 3. *If a collective choice rule satisfies independence of irrelevant alternatives, the Pareto condition, and base quasi-transitivity, and if i is weakly decisive for some $x, y \in X$, then i is a weak dictator; that is,*

$$\begin{aligned}&(xP_iy \;\&\; yP_jx \textit{ for all } j \neq i \Rightarrow x \in C(\{x, y\}))\\ &\quad \Rightarrow (\textit{For all } s, t \in X : sP_it \Rightarrow s \in C(\{s, t\})).\end{aligned}$$

The proof of this lemma is virtually identical to that of Lemma 2 and is omitted.

Lemma 4. *if V is a smallest decisive set with respect to a and b under a collective choice rule satisfying base quasi-transitivity, the Pareto condition, nondictatorship, and independence of irrelevant alternatives, then*

$$V \textit{ contains at least two individuals,} \tag{3}$$
$$\textit{and every } i \in V \textit{ is a weak dictator.} \tag{4}$$

Proof. Assertion (3) is obvious from Lemma 2 and nondictatorship. To establish (4), we must show:

$$\textit{If } i \in V, xP_iy \Rightarrow x \in C(\{x, y\}) \textit{ for some } x, y \in X.$$

Suppose not. Then for some $a, z \in X$, there exists a profile of the form:

$i :$ $a, z,$
$N \setminus \{i\}$: (some $n - 1$-tuple of orderings of a and z).

such that $\{z\} = C(\{a, z\})$. Let $W \subset V$ and $V \setminus W = \{i\}$. Consider the following further specification of the previous profile:

i : $a, b, z,$
W: (same individual orderings as before between a and z), b,
$N \setminus V$: b, (same individual orderings as before between a and z).

We know $C(\{a, z\}) = \{z\}$ and, because of V's decisiveness for a over b, $C(\{a, b\}) = \{a\}$. By base quasi-transitivity, $\{z\} = C(\{b, z\})$ under the profile in question. But this implies that W is decisive for z against b, which contradicts the minimality of V. Thus, if $i \in V$, i is weakly decisive for some x against $y, x, y \in X$. By Lemma 3, such an individual is a weak dictator, establishing (4).

Proof of Theorem 4. Every dictator is a weak dictator, so prohibiting weak dictators rules out dictators too. If a collective choice rule satisfies base quasi-transitivity, independence of irrelevant alternatives, the Pareto condition, and has no weak dictators (and thus no dictator either), then Lemma 4 yields the conclusion that there must exist a weak dictator, which contradiction proves the theorem.

Mas-Colell and Sonnenschein's [14] result on the inconsistency of quasi-transitivity and positive responsiveness in the presence of the Arrovian conditions carries over in a straightforward manner to the case of irrational path independence, and thence to base quasi-transitivity:

Theorem 5. *If there are at least three voters, there is no collective choice rule satisfying all of*:

(1) *base quasi-transitivity,*
(2) *the Pareto condtion,*
(3) *nondictatorship,*
(4) *independence of irrelevant alternatives, and*
(5) *positive responsiveness.*

Proof. By Lemma 4, there exist at least two weak dictators; call them 1,2. Suppose under some profile xP_1y and yP_2x for some $x, y \in X$; by weak dictatorship, $\{x, y\} = C(\{x, y\})$, regardless of the preferences of others voters, of whom there exists at least one. This violates positive responsiveness.

In view of the impossibility theorems discussed in the Introduction and the new results just presented, one might be tempted to retreat and demand the imposition only of the Chernoff condition which, as we have seen, is strictly weaker than both acyclicity and path independence. This condition is an appealing one to impose on collective choice rules. It is clearly desirable in piecemeal choice mechanisms where choices are made from unions of choices over subsets. If an alternative fails to be chosen in some subset, it need not be considered again at a later stage, for the contra-positive of the Chernoff condition ensures that the alternative will not be among the final choices. Arrow's justification in [2, pp.26-27] for his independence axiom is obviously instead an argument for this condition. See also Sen [23, p.17]. Nevertheless, as the following theorems demonstrate, the Chernoff condition standing alone as a rationality condition must also be rejected, at least if the Arrovian conditions are found compelling.[4]

Theorem 6. *If there are at least four voters, there is no collective choice rule satisfying all of:*

(1) *the Chernoff condition,*
(2) *the Pareto condition,*
(3) *non-weak-dictatorship,*
(4) *independence of irrelevant alternatives, and*
(5) *positive responsiveness.*

As in the case of Theorem 3, we will establish this result by proving an even stronger proposition. By Lemma 1, the Chernoff condition implies base acyclicity, and it is obvious from the definitions that base acyclicity implies base triple acyclicity.

Theorem 7. *If there are at least four voters, there is no collective choice rule satisfying all of:*

(1) *base triple acyclicity,*
(2) *the Pareto condtion,*
(3) *non-weak-dictatorship,*
(4) *independence of irrelevant alternatives, and*
(5) *positive responsiveness.*

Proof. We will proceed in two steps. First, assuming all of the conditions in the theorem except non-weak-dictatorship, we will show that there exists a voter who is weakly decisive for some pair of alternatives. We will then show that individual is a weak dictator, contradicting the third condition in the theorem.

Step 1. We must show that if a collective choice rule satisfies base triple acyclicity, the Pareto condition, independence, and positive responsiveness, there exists an individual $i \in N$ and alternatives $x, y \in X$ such that:

$$(xP_iy \ \& \ yP_jx \ for \ all \ j \neq i) \Rightarrow x \in C(\{x, y\}). \tag{5}$$

[4]The existence of collective choice rules which satisfy the Chernoff condition but are neither path-independent nor rational is guaranteed by the following proposition, which is easily verified: the rule which makes the collective choice equal to the union of individuals' choices from the feasible set satisfies the Chernoff condition, if the individuals' preferences satisfy that condition as well. The group's choice function is precisely the one given in the example in Section 2 if the group has two members with the following acyclic preferences: $xP_1y, yP_1z, xI_1z, xP_2z, zP_2y, xI_2y$, and the collective choice rule is the union-rule just described.

Suppose (5) is false. Then for all $x, y \in X$ and all $i \in N$, if a profile a is such that under it xP_iy and yP_jx for all $j \in N \setminus \{i\}$, then $C^a(\{x, y\}) = \{y\}$. Let V be a smallest decisive set; V is decisive for some x against y. Our assumption implies that V contains at least two individuals, say 1 and 2. Partition V as $V = \{1, 2\} \cup V^*$. Consider profile b:

$$\begin{array}{ll} 1: & x, y, z \\ \{2\} \cup V^*: & z, x, y \\ N \setminus V: & y, z, x. \end{array}$$

By the definition of V, $\{x\} = C^b(\{x, y\})$, and by assumption, $C^b(\{x, z\}) = \{z\}$. By base triple acyclicity, $z \in C^b(\{y, z\})$. Now consider profile c:

$$\begin{array}{ll} 1: & x, y \\ 2: & y, x \\ V^*: & x, y \\ N \setminus V: & y, x. \end{array}$$

Since V is a smallest decisive set, $y \in C^c(\{x, y\})$. Next consider profile d:

$$\begin{array}{ll} 1: & (x, y, z) \\ 2: & z, y, x \\ V^*: & x, z, y \\ N \setminus V: & y, x, z. \end{array}$$

Comparing profiles, c and d, and noting the conclusion drawn from the former, positive responsiveness and independence require that $\{y\} = C^d(\{x, y\})$. Comparing profiles b and d, the same two axioms require that $\{z\} = C^d(\{y, z\})$. Base triple acyclicity then yields $z \in C^d(\{x, z\})$. Next examine profile e:

$$\begin{array}{ll} 1: & z, x \\ 2: & z, x \\ V^*: & x, z \\ N \setminus V: & x, z. \end{array}$$

Comparing profiles d and e, positive responsiveness and independence require that $\{z\} = C^e(\{x, z\})$. This conclusion and independence imply that $\{1, 2\}$ is decisive for z against x, so that $V = \{1, 2\}$. Finally examine profile f:

$$\begin{array}{ll} 1: & x, y, z \\ 2: & z, x, y \\ N \setminus V: & y, z, x. \end{array}$$

Since V is decisive for x against y, $\{x\} = C^f(\{x, y\})$, while our assumption yields $\{y\} = C^f(\{y, z\})$. By base triple acyclicity, $x \in C^f(\{x, z\})$, in contradiction to our assumption. Thus we have shown that voter 1 is weakly decisive for x against z.

Step 2. In this step we show that if voter 1 is weakly decisive for x against y, then he or she is a weak dictator, if there are at least four voters. In the presence of positive responsiveness, this can be established by proving that for all $s, t \in X$,

$$(sP_1t \ \& \ tP_js \text{ for all } j \in N \setminus \{1\}) \Rightarrow s \in C(\{s, t\}). \tag{6}$$

We will prove only that for all $t \in X$,

$$(xP_1t \ \& \ tP_jx \text{ for all } j \in N \setminus \{1\}) \Rightarrow s \in C(\{x, t\}). \tag{7}$$

The steps from (7) to (6) are sufficiently similar to the ones we use in establishing (7) that they may safely be skipped. To prove (7), we first examine profile a:

$$\begin{array}{ll}
1: & x, y, t \\
2: & (x, y), t \\
3: & y, t, x \\
4: & y, t, x \\
N \setminus \{1, 2, 3, 4\}: & y, t, x.
\end{array}$$

By Step 1 and positive responsiveness, $C^a(\{x, y\}) = \{x\}$. By the Pareto condition, $C^a(\{y, t\}) = \{y\}$. By base triple acyclicity, $x \in C^a(\{x, t\})$. Now consider profile b:

$$\begin{array}{ll}
1: & y, x, t \\
2: & y, x, t \\
3: & t, y, x \\
4: & y, (x, t) \\
N \setminus \{1, 2, 3, 4\}: & t, y, x.
\end{array}$$

Comparing profiles a and b, positive responsiveness and independence require that $C^b(\{x, t\}) = \{x\}$. By the Pareto condition, $C^b(\{x, y\}) = \{y\}$. Base triple acyclicity then implies $y \in C^b(\{y, t\})$. Next examine profile c:

$$\begin{array}{ll}
1: & x, y, t \\
2: & y, t, x \\
3: & (x, y, t) \\
4: & y, t, x \\
N \setminus \{1, 2, 3, 4\}: & t, y, x.
\end{array}$$

By Step 1 and positive responsiveness, $C^c(\{x, y\}) = \{x\}$. Comparing profiles b and c, positive responsiveness and independence require that $C^c(\{y, t\}) = \{y\}$. Another application of base triple acyclicity yields $x \in C^c(\{x, t\})$. Next consider profile d:

$$\begin{array}{ll}
1: & y, x, t \\
2: & t, y, x \\
3: & y, x, t \\
4: & t, y, x
\end{array}$$

$$N \setminus \{1,2,3,4\}: \quad t, y, x.$$

Comparing profiles c and d, positive responsiveness and independence require $C^d(\{x,t\}) = \{x\}$; by the Pareto condition $\{y\} = C^d(\{x,y\})$. Base triple acyclicity then yields $y \in C^d(\{y,t\})$. Finally consider profile e:

$$\begin{array}{ll} 1: & x, y, t \\ 2: & t, (x, y) \\ 3: & y, t, x \\ 4: & (y, t), x \\ N \setminus \{1,2,3,4\}: & t, y, x. \end{array}$$

Comparing profiles d and e, positive responsiveness and independence again require $\{y\} = C^e(\{y,t\})$. By Step 1 and positive responsiveness, $C^e(\{x,y\}) = \{x\}$. A final application of base triple acyclicity yields $x \in C^e(\{x,t\})$. In view of the independence axiom, this proves (7).

4 Concluding Remarks

Arrow and subsequent writers have modeled the output of collective decision-making institutions as binary social preference relations, both by analogy with consumers' preferences in demand theory and as a generalization of Condorcet's proposal that any alternative which received a majority of votes against every other candidate should be chosen. Such a view, as the well-known series of impossibility theorems demonstrates, is inconsistent with a set of several democratic requirements. In this chapter we have shown that binary rationality per se is not the culprit in these theorems.

We have taken a more general view, and required only that the group makes a nonempty choice from every finite feasible set of alternatives. Several weak conditions imposed on the resultant choice of functions are each shown to contradict one or more of the same democratic requirements, even if the choices have no binary rationalization.

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Chapter 10
Remarks on the Theory of Collective Choice*

Ever since the so-called paradox of voting was generalized by Arrow [2] to every democratic method of collective decision-making, a vast literature has appeared (a) trying to circumvent Arrow's difficulty by weakening some of his conditions (Bordes [6]; Hansson [12]; Plott [15]; Sen [20]); (b) proposing some other paradoxes in the theory of collective choice (Batra and Pattanaik [3]; Hansson [11]; Schwarts [19]; Sen [21]) and (c) casting doubts about the relevance of Arrow's theorem to the theory of Paretian welfare economics (Bergson [4]; Little [14], Samuelson [17; 18]). The purpose of this chapter is to make some remarks on these recent developments in the theory of collective choice.

The first part of the chapter deals with the question of how much one needs to weaken Arrow's collective rationality condition in order to avoid his impossibility result. As is well known, Arrow [2] imposed the collective rationality condition that the society can arrange all conceivable alternatives in order of preference and that, if some available set of alternatives is specified, the society must choose therefrom the best alternative(s) with respect to that preference ordering. We will consider two conditions of consistent choice which are weaker than that of Arrow. The first condition requires that, if an alternative x is chosen over another alternative y in binary choice, y should never be chosen from any set of alternatives that contains x; while the second condition requires that, if x is chosen over y in binary choice, there exists no choice situation in which y is chosen and x is available but rejected. (In the second case y can be chosen if x is also chosen, while in the first case y cannot be chosen anyway.) There seems to be a gulf that separates possibility from impossibility in between these two seemingly similar consistency conditions. It will be shown that the first consistency requirement is *incompatible* with essentially Arrovian conditions on the collective choice rule, while the second consistency condition is *compatible* with the same conditions. Although the difference between these consistency requirements is very subtle, the implication thereof in the context of impossibility result is therefore dramatically different.

Lest we should be too satisfied, we must hasten to add that no collective choice rule satisfying our second consistency requirement can be free from the paradox of a Paretian liberal (Sen [21]; Batra and Pattanaik [3]).

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*First published in *Economica*, Vol.43, 1976, pp.381-390. Thanks are due to Professors A. K. Sen and G. Bordes for their comments. They are, of course, not responsible for any remaining deficiencies of this chapter. Thanks are also due to an anonymous referee and the editor of *Economica*.

The Arrow-Sen theory is then contrasted with the Bergson-Samuelson one. In view of doing this, it is convenient to remember that Arrow's incompatible conditions on collective choice rule can be classified into two categories. The first category consists of statements that apply to any *fixed* profile of individual preference relations, while the second category refers to the responsiveness of the collective choice to the *variations* in profiles. (The first category embraces the condition of *collective rationality* and the *Pareto rule*, while the second category consists of the *independence of irrelevant alternatives* and *non-dictatorship*. See Sections I and II below for the definition of these conditions.) Begson [4], Little [14] and Samuelson [17; 18] agreed with Arrow so far as his conditions of the first category were concerned. It was only when Arrow went on to introduce some conditions of the second category that Bergson, Little and Samuelson come to deny the reasonableness and/or necessity thereof. Let us, therefore, fix a profile of individual preference relations. What we call the *Bergson-Samuelson social welfare ordering* is an ordering R such that, if a state x is Pareto-non-inferior (resp. Pareto-superior) to a state y with respect to the given profile, then x is not less preferred (resp. preferred) to y in terms of R. (Incidentally, the Bergson-Samuleson social welfare function is a numerical function u such that $u(x) \geq u(y)$ if and only if xRy for all states x and y.) Now we raise the problem of whether we can define such a social welfare ordering corresponding to a given profile. The answer is in the affirmative if each and every individual preference relation satisfies a strong consistency condition of transitivity. The main purpose in this part is to strengthen this in that a social welfare ordering exists if and only if the Paretian unanimity rule corresponding to the given profile satisfies what we call the *axiom of consistency*. (Thus, even if intransitive individual preference is countenanced, we may still have a well-defined social welfare ordering.) This will be done by proving a general theorem on the extension of a binary relation.

What emerges as a result of our investigation is an ever sharper contrast between the *variable* profile framework of the Arrow-Sen theory, on the one hand, and the *fixed* profile framework of the Bergson-Little-Samuelson theory, on the other.

All the proofs are relegated into the final section which can be neglected by those who are not interested in technical details.

1 Rationality and Revealed Preference

For the sake of logical clarity, we discuss in this section the concept of rationality and that of revealed preference in abstraction from the problem of collective decision. The heart of our argument is the implication diagram given at the end of this section.

Let X be the set of all alternatives that are mutually exclusive. We assume that X contains at least three distinct elements. Also let K stand for the family of non-empty subsets of X, containing all the pairs and all the triples taken from X (and possibly more). A preference relation is a binary relation on X. Let R be a binary relation. By xRy (or, equivalently, $(x, y) \in R$) we mean that x is at least as good as y. By xPy and xIy we mean, respectively, (xRy and *not* yRx) and (xRy and yRx). An R is said to be *reflexive* if (xRx for all $x \in X$), *complete* if (xRy and/or yRx for all $x, y \in X$). *transitive* if (xRy and $yRz \Rightarrow xRz$ for all $x, y, z \in X$), and *acyclic* if ($x^1Px^2 \dots x^tPx^1$ for no finite subset $\{x^1, x^2, \dots, x^t\}$ of X). A *quasi-ordering* is a reflexive and transitive relation and an *ordering* is a complete quasi-ordering.

Given a preference relation R and an $S \in K$, we define:

$$(1) \qquad G_R(S) = \{x : x \in S \text{ and } xRy \text{ for all } y \in S\}.$$

Clearly $G_R(S)$ is a subset of S such that $x^* \in G_R(S)$ has the following property: x^* is at least as good as any alternative in S.

A *choice functioin* on K is a function C which assigns a non-empty subset $C(S)$ of S to each $S \in K$. (It is intended that $S \in K$ represents a set of available alternatives and $C(S)$ represents the set of chosen elements from S.) We say that C is *rational* (R) if there exists a preference relation R, to be called a *rationalization* of C, such that

$$(2) \qquad C(S) = G_R(S) \text{ for all } S \in K.$$

In other words, a choice function is rational if it can be construed as a result of preference optimization. (It should be noted that the concept of rational choice in itself has noting to do with the transitivity of rationalization.) We say, in particular, that C is *regular-rational* (RR) if (a) C is rational and (b) a rationalization thereof is an ordering. Arrow [1] has shown that C is (RR) if and only if, for all S_1 and S_2 in K such that $S_1 \subset S_2$ and $S_1 \cap C(S_2) \neq \phi$, $S_1 \cap C(S_2) = C(S_1)$ holds true. (In other words, it is required that if some elements are chosen out of S_2, and then the range of alternatives is narrowed to S_1 but still contains some previously chosen elements, no previously rejected element becomes chosen and no previously chosen element becomes unchosen.) This *Arrovian property* (A) is to be decomposed into what we call the *Bordesian property* (B) and the *Chernovian property* (C). We say that C satisfies (B) (resp. (C)) if and only if, for all S_1 and S_2 in K such that $S_1 \subset S_2$ and $S_1 \cap C(S_2) \neq \phi$, $S_1 \cap C(S_2) \supset C(S_1)$ (resp. $S_1 \cap C(S_2) \subset C(S_1)$). The property (B) requires that if some elements are chosen out of S_2 and then the range of alternatives is narrowed to S_1 but still contains some previously chosen elements, no previously unchosen element becomes chosen. The property (C) requires, on the other hand, that if some elements of subset S_1 of S_2 are chosen from S_2, then they should be chosen from S_1. The property (B) is due to Bordes [6], while the property (C) is named after Chernoff [8], although in the present context it is better known as Sen's condition α (see Sen, [20; 22]).

An alternative formulation of the concept of rational choice goes as follows. Given a preference relation R and an $S \in K$, we define:

$$M_R(S) = \{x : x \in S \text{ and } \textit{not } yPx \text{ for all } y \in S\}.$$

We say that a choice function C is *M-rational* if there exists a preference relation R, to be called an *M-rationalization*, such that $C(S) = M_R(S)$ for all $S \in K$. In view of some arguments by Herzberger [13] and Schwartz [19] in favour of the M-rationality concept, it may be worth our while to investigate how M-rationality will fare in the context of the impossibility results. In order to do so, let C be M-rational with an M-rationalization R.

Let us define a binary relation R' by $\{xR'y \Leftrightarrow (xRy \text{ or } \textit{not } yRx)\}$ for all x and y in X. It can easily be shown that $M_R(S) = G_{R'}(S)$ for all $S \in K$. Therefore *if C is M-rational, it is rational*. This being the case, the concept of M-rationality has no special role to play in the impossibility exercises (see, however, Suzumura [24]).

So much for rational choice functions. Let us now introduce some axioms of revealed preference. Our *first axiom of revealed preference* (FARP) and *second axiom of revealed preference* (SARP) consider the binary choice of x over $y : \{x\} = C(\{x, y\})$. In this case, (FARP) requires that there should be no choice situation $S \in K$ such that $x \in S$ and $y \in C(S)$, while (SARP) requires that there should be no choice situation $S \in K$ such that $x \in S \setminus C(S)$ and $y \in C(S)$. What these two axioms essentially say is that the choice pattern revealed in binary choice should never be contradicted in non-binary choice. Our third revealed preference axiom, which we called in Blair *et al.* [5] the *base triple acyclicity* (BTA), is concerned solely with binary choices. It requires that if x is chosen over y in binary choice and y is chosen over z in binary choice, x should never be rejected in the binary choice between x and $z : \{x\} = C(\{x, y\})$ and $\{y\} = C(\{y, z\}) \Rightarrow x \in C(\{x, z\})$ for all $x, y, z \in X$. Finally we introduce the *weak axiom of revealed preference* (WARP), due originally to Samuelson [16], who introduced it in the context of consumers' behaviour. It says that, if $\{x \in C(S)$ and $y \in S \setminus C(S)\}$ for some $S \in K$, then $\{x \in S'$ and $y \in C(S')\}$ for no $S' \in K$. Namely, if in some choice situation x is chosen while y, though available, is rejected, then y should never be chosen in the presence of x.

Essential for our present purpose is an implication network among these various requirements on the choice function, which is summarized in the implication diagram of Figure 1. Here, a double-headed arrow indicates equivalence, and a single-headed arrow indicates implication. Some of these arrows have been established by Arrow [1] and Sen [20], while the remaining ones will be proved in Section V.

$$\begin{tikzcd}
 & \text{(R)} \arrow[rr] & & \text{(C)} \arrow[d] & \\
\text{(A)} \arrow[r, leftrightarrow] & \text{(RR)} \arrow[u] \arrow[d] \arrow[r, leftrightarrow] & \text{(WARP)} \arrow[r] & \text{(FARP)} \arrow[r] \arrow[d] & \text{(BTA)} \\
 & \text{(B)} \arrow[rr] & & \text{(SARP)} &
\end{tikzcd}$$

Figure I

2 Arrow's Theorem and Collective Rationality

We are ready to discuss the problem originally posed by Arrow [2]. Suppose that there exist n individuals in the society and let $N = \{1, \ldots, i, \ldots, n\}$ stand for the index set of the individuals. In order to exclude a trivial case, we assume that $n \geq 2$. An n-tuple of preference orderings $(R_1, \ldots, R_i, \ldots, R_n)$, one ordering for each individual, will be called a *profile* (of individual preference orderings). Corresponding to R_i, we define P_i

by $xP_iy \Leftrightarrow (xR_iy$ and *not* $yR_ix)$. The problem of collective choice is to find a function F, to be called a *collective choice rule* (CCR), which aggregates a profile into a collective choice function:

$$C = F(R_1, \ldots, R_i, \ldots, R_n). \tag{3}$$

If an $S \in K$ is specified as an available alternatives set, $C(S)$ represents the set of socially chosen elements from S when the profile $(R_1, \ldots, R_i \ldots, R_n)$ prevails. In what follows, we will always assume *universal domain* (U) for F: F should be able to aggregate all logically possible profiles. As a matter of terminology, we say that F satisfies the property ω if F always yields a choice function having the property ω. (For example, if C determined by (3) is always rational, we say that F is rational in itself.) Some other conditions on CCR will be introduced when and where necessity dictates.

Arrow [2] has shown that *there exists no* CCR *that satisfies the universal domain* (U), *the regular-rationality* (RR), *the independence of irrelevant alternatives* (I), *the Pareto rule* (P) *and the non-dictatorship* (D). Here the conditions (I), (P) and (D) are defined as follows. Let x and y be any two alternatives. A CCR F is said to satisfy the condition (I) if, for any two profiles $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$ such that $(xR_iy \Leftrightarrow xR'_iy$ and $yR_ix \Leftrightarrow yR'_ix)$ for all $i \in N$, we have $C(\{x, y\}) = C'(\{x, y\})$; the condition (P) if, for any profile $(R_1, \ldots, R_n)$ such that xP_iy for all $i \in N$, we have $\{x\} = C(\{x, y\})$; the condition (D) if there exists no individual $i \in N$ such that, for any profile $(R_1, \ldots, R_n)$, $xP_iy \Rightarrow \{x\} = C(\{x, y\})$. Throughout in what follows we let $C = F(R_1, \ldots, R_n)$, and $C' = F(R'_1, \ldots, R'_n)$.

Among these Arrow's incompatible conditions on CCR, (P) and (D) can hardly be objectionable, so that the culprit for Arrow's phantom should be sought among (U), (RR) and (I). In what follows, our attention will be focused upon the condition (RR). (The relevance of the condition (U) is extensively discussed in Sen [21, Chapter 10] in the context of simple majority decision, while condition (I) is critically examined by Hansson [12].) Can we circumvent Arrow's difficulty by weakening (RR) to some reasonable extent?

In order to prepare for our answer to this question, it is necessary to introduce some more conditions on CCR. First, there is the condition of the *non-weak-dictatorship* (WD). Let x and y be any two alternatives. We say that F satisfies the condition (WD) if there exists no $i \in N$ such that, for any profile $(R_1, \ldots, R_n), xP_iy \Rightarrow x \in C(\{x, y\})$. Clearly (WD) is a stronger version of (D). Second, there is the condition of the *positive responsiveness* (PR). Let $i \in N$ be any prescribed individual and let x and y be any two alternatives. We say that F satisfies the condition (PR) if (a) $(R_1, \ldots, R_n)$ is a profile resulting in $x \in C(\{x, y\})$, and (b) $(R'_1, \ldots, R'_n)$ is another profile with $R_j = R'_j$ for all $j \in N \setminus \{i\}$ and $\{(yP_ix$ and $xI'_iy)$ or $(xI_iy$ and $xP'_iy)\}$, then $\{x\} = C'(\{x, y\})$.

We now put forward two theorems which are relevant in the context of the question raised in this section.

Theorem 1. *If* $n \geq 4$, *there exists no* CCR *which satisfies* (U), (FARP), (P), (I), (WD) *and* (PR).

Theorem 2. *There exists a* CCR *which satisfies* (U), (SARP), (P), (I), (WD) *and* (PR).

Note that the impossibility in Theorem 1 is turned into the possibility in Theorem 2 by simply replacing the condition (FARP) by the condition (SARP). Put differently, in the presence of (U), (P), (I), (WD) and (PR), the gulf that separates possibility from impossibility is located by Theorem 1 and Theorem 2 as being in between (FARP) and (SARP).

3 Paradise Lost

It is now time to make some observations by comparing Arrow's theorem, Theorem 1 and Theorem 2. First, let us compare Theorem 1 with Arrow's theorem. In Theorem 1, Arrow's rationality condition (RR) is substantially weakened into (FARP), but (D) is strengthened into (WD), and (PR) (which does not appear in Arrow's theorem) is invoked. Therefore, strictly speaking, Theorem 1 is not a generalization of Arrow's theorem. It may, however, be claimed that (WD) and (PR) are still reasonable conditions on a democratic collective choice rule and our Theorem 1 may be taken to mean that Arrow's difficulty cannot be got rid of even if his rationality condition (RR) is substantially weakened.

Let us next compare Theorem 1 with Theorem 2. We have noticed already that, although (FARP) and (SARP) look quite similar, their implications in the context of collective choice are very disparate. The contrast being sharp, we might be tempted to say that the Arrovian impossibility depended squarely on the unjustifiably strong collective rationality requirement such as (FARP) and that if we replaced (FARP) by a weaker (SARP), the Arrovian phantom would go. Life would be happier then for democrats if this could really be the end of the story. Unfortunately, however, this is not the case. The gist is that the theory of collective choice is full of disturbing paradoxes and Arrow's theorem is only one eminent example.

Two more conditions on the CCR are to be introduced. The first one is a strengthened version of the Pareto rule (P). Let S be any set in K and let $(R_1, \ldots, R_n)$ be any profile. Let y be any point in S. We say that F satisfies the *strong Pareto rule* (SP) if $\{(xP_iy \text{ for all } i \in N) \text{ for some } x \in S\} \Rightarrow y \notin C(S)$. In words, y should not be chosen out of S if there exists an x in S that is unanimously preferred to y. The second condition is what Sen [21] called the condition of *minimal liberalism* (ML), which reads as follows. There are at least two individuals such that for each of them there is at least one pair of alternatives over which he is decisive; that is, there is a pair of x, y such that if he prefers x (respectively y) to y (respectively x), then society should prefer x (respectively y) to y (respectively x) (Sen [21, p.154]).

Sen [21] has shown that *there exists no* CCR *that satisfies universal domain* (U), *rationality* (R), *the Pareto rule* (P) *and minimal liberalism* (ML). This so-called *liberal paradox* has been generalized by Batra and Pattanaik [3], from which it follows that *there exists no* CCR *that satisfies* (U), (SARP), (SP) *and* (ML). This being the case, we cannot but say that, although the replacement of (FARP) by (SARP) fares quite well in exorcising the Arrovian phantom, it cannot let the CCR be free from Sen's liberal paradox. The cloud is thicker here because the independence condition (I) (which has also been suspected to be a possible culprit for Arrow's difficulty) does not play any role at all in establishing Sen's paradox. Our conclusion is that, even if the collective rationality condition is substantially weakened, we cannot eradicate the paradoxes of collective decision.

4 Bergson-Samuelson Social Welfare Ordering

We now turn to discuss the logical foundation of the Bergson-Samuelson theory of Paretian welfare economics. It has long been lamented that Arrow gave his collective choice rule, which is, in our terminology, a regular-rational CCR, the name of *social welfare function*. Clearly it is completely different from the Bergson-Samuelson social welfare function which, according to Little, is "a 'process or rule' which would indicate the best economic state as a function of a changing environment (i.e. changing sets of possibilities defined by different economic transformation functions), *the individual tastes being given*" (Little [14, p.423], Little's italics). It is also claimed that "the only axiom restricting Bergson social welfare function (of individualistic type) is a 'tree' property of Pareto-optimality type" (Samuelson [17, p.49]). The purpose of the rest of this chapter is to examine the possibility of this *fixed* profile theory of Paretian welfare economics.

Let us, therefore, fix a profile $(R_1, \ldots, R_i, \ldots, R_n)$ of the individual preference relations. Let a binary relation Q, to be called the *Pareto unanimity relation*, be defined by $(xQy \Leftrightarrow xR_iy$ for all $i \in N)$ for all $x, y, \in X$. The asymmetric component P_Q and the symmetric component I_Q of Q are defined, respectively, by $(xP_Qy \Leftrightarrow xQy$ and *not* $yQx)$ and $(xI_Qy \Leftrightarrow xQy$ and $yQx)$.

A *social welfare ordering* (SWO) in the sense of Bergson and Samelson is an ordering R such that $\{(xQy \Rightarrow xRy)$ and $(xP_Qy \Rightarrow (xRy$ and *not* $yRx))\}$ for all $x, y \in X$. (In words, an SWO is an ordering that preserves whatever information the Pareto unanimity relation can tell us about the wishes of the individuals.) A *social welfare function* (SWF) in the sense of Bergson and Samuelson is a numerical representation u of $R : u(x) \geq u(y) \Leftrightarrow xRy$.

Our problem is to examine the existence of an SWO for a fixed profile. Thanks to the work of Debreu [9] and others, we know that an SWO may not have an SWF representing it. It may, however, be said that what is important is an SWO but not its numerical representation.

Suppose that $R_1, \ldots, R_n$ are orderings. In this case Q is a quasi-ordering, so that a corollary of Szpilrajn's theorem (Fishburn [10, Lemma 15.4]) assures us of the existence of an SWO corresponding to the given profile. How about the case where R_i $(i \in N)$ is not necessarily transitive? Generally speaking, we cannot have an SWO in this case, as the following examples where $n = 3$ and $X = \{x, y, z\}$ exhibit.

$$\text{(A)}\ \begin{array}{l} xI_1y, yI_1z, zP_1x \\ yI_2z, zI_2x, xP_2y \end{array} \qquad \text{(B)}\ \begin{array}{l} xP_1y, yI_1z, zI_1x \\ yI_2z, zI_2x, xP_2y \end{array}$$

$zI_3x, xI_3y, yP_3z \qquad\qquad zI_3x, xI_3y, yP_3z$

(Notice here that in both profiles individual strict preference is transitive but individual indifference is not.) In the case (A), we have xP_Qy, yP_Qz and zP_Qx, so that Q is cyclic and there exists no ordering which can subsume this Q. In the case (B), we have xP_Qy, yP_Qz and xI_Qz. Although this Q is acyclic, we cannot still have an ordering subsuming it. Under what condition can Q have an ordering which subsumes it?

Our answer will be given via a general theorem on the extension of a binary relation. Let R be a given binary relation. A t-tuple of alternatives $(x^1, x^2, \ldots, x^t)$ is called a *PR-cycle* of order t if we have $x^1Px^2R\ldots Rx^tRx^1$, where P is the asymmetric component of R. We say that R is *consistent* if there exists no *PR-cycle* of any finite order. It is clear that a consistent binary relation is acyclic but not vice versa. An ordering R^* is said to be an *extended ordering* of R if $\{(xRy \Rightarrow xR^*y)$ and $(xPy \Rightarrow xP^*y)\}$ for all $x, y \in X$, where P^* is the asymmetric component of R^*. We can now state a theorem on the existence of extended ordering.

Theorem 3. *A binary relation R has an extended ordering R^* if and only if R is consistent.*

In passing we note that Champernowne [7] introduced a concept of consistent preference (or probability) relations which is similar to but distinct from ours. We say that a t-tuple $(x^1, x^2, \ldots, x^t)$ is a *C-cycle* of order t if $x^1(P \cup N)x^2R\ldots Rx^tRx^1$, where $\{x^1(P\cup N)x^2 \Leftrightarrow (x^1Px^2 \text{ or } x^1Nx^2)\}, \{x^1Nx^2 \Leftrightarrow (\textit{not}\ x^1Rx^2 \text{ and } \textit{not}\ x^2Rx^1)\}$. R is said to be *Champernowne-consistent* if there exists no *C-cycle* of any finite order. Unfortunately it turns out that R is Champernowne-consistent if and only if it is transitive. Clearly we have only to show that Champernowne consistency implies transitivity. Suppose that R is not transitive. Then we have xRy, yRz but not xRz for some $x, y, z \in X$. Therefore we have $z(P \cup N)xRyRz$, so that (z, x, y) is a *C-cycle* of order 3. In order to show that our concept of consistency does not reduce to transitivity, we give an example. Let $X = \{x, y, z\}$ and let R be defined by xPy, yRz and xNz.

It follows from Theorem 3 that a social welfare ordering exists for a profile $(R_1, \ldots, R_n)$ if and only if the Pareto unanimity relation Q corresponding to this profile is consistent. In this context, it is worth our while to note that the transitivity of an R_i implies that of strict preference P_i and of indifference I_i, and cases against transitive indifference are plenty. This being the case, it is interesting to see that we can define an SWO corresponding to a profile $(R_1, \ldots, R_n)$ even if R_i is not necessarily transitive so far as Q satisfies the axiom of consistency. The contrast between the *variable* profile framework of the Arrow-Sen theory, which leads us to the logical impossibility even under weakened rationality requirement, and the fixed profile framework of the Bergson-Samuelson theory, which can even accommodate some individual preference intransitivity, is made sharper than ever.

5 Proofs

Proof of the Implication Diagram

First, we show that (C) implies (FARP). Let x and y be such that $C(\{x, y\}) = \{x\}$ and take a superset $S \in K$ of $\{x, y\}$. By virtue of (C), we have $\{x, y\} \cap C(S) \subset C(\{x, y\})$, so that $y \notin C(S)$.

Second, we show that (FARP) implies (BTA). If (BTA) is not satisfied by C, there exist x, y and z in X such that $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$ and $C(\{x, z\}) = \{z\}$. Suppose that C satisfies (FARP). Then x does not belong to $C(\{x, y, z\})$, because $\{z\} = C(\{x, z\})$. Similarly, neither y nor z belong to $C(\{x, y, z\})$. Thus $C(\{x, y, z\}) = \emptyset$, a contradiction. Therefore (FARP) implies (BTA).

Third, it will be shown that (B) implies (SARP). If C does not satisfy (SARP), there exist $x, y \in X$ and $S \in K$ such that $C(\{x, y\}) = \{x\}$, $x \in S \setminus C(S)$ and $y \in C(S)$. Let $S' = \{x, y\}$. Then $S' \subset S$, $S' \cap C(S) \neq \emptyset$, but $x \notin S' \cap C(S)$ and $x \in C(S')$, so that (B) does not hold. Therefore, (B) implies (SARP).

In order to show that a single-headed arrow in the diagram cannot in general be reversed, we put forward the following examples.

Example 1. $X = \{x, y, z\}, K = \{S_1, \ldots, S_7\}, S_1 = \{x\}, S_2 = \{y\}, S_3 = \{z\}, S_4 = \{x, y\}, S_5 = \{y, z\}, S_6 = \{x, z\}, S_7 = X, C(S_t) = S_t\ (t = 1, 2, 3), C(S_4) = S_1, C(S_5) = S_2, C(S_6) = S_3$ and $C(S_7) = S_7$.

Example 2. The same as Example 1 except for $C(S_6) = S_6$ and $C(S_7) = S_2$.

Example 3. The same as Example 1 except for $C(S_6) = S_6$ and $C(S_7) = S_1$.

Example 4. The same as Example 1 except for $C(S_6) = S_1$.

Example 5. The same as Example 1 except for $C(S_4) = S_4$, $C(S_5) = S_5$ and $C(S_6) = S_6$ and $C(S_7) = S_1$.

Example 6. $X = \{w, x, y, z\}, K = \{S_1, \ldots, S_{15}\}, S_1 = \{x\}, S_2 = \{y\}, S_3 = \{z\}, S_4 = \{w\}, S_5 = \{w, x\}, S_6 = \{w, y\}, S_7 = \{w, z\}, S_8 = \{x, y\}, S_9 = \{x, z\}, S_{10} = \{y, z\}, S_{11} = \{w, x, y\}, S_{12} = \{w, x, z\}, S_{13} = \{w, y, z\}, S_{14} = \{x, y, z\}, S_{15} = X, C(S_t) = S_t\ (t = 1, \ldots, 10), C(S_{11}) = S_8, C(S_{12}) = S_1, C(S_{13}) = S_2, C(S_{14}) = S_1$ and $C(S_{15}) = S_2$.

The choice function in Example 1 satisfies (SARP) but not (FARP), so that (SARP) does not necessarily imply (FARP). The choice function in Example 2 satisfies (BTA) but not (FARP), so that (BTA) does not imply (FARP) in general. (Incidentally, Example 1 gives a choice function which satisfies (SARP) but not (BTA), while the choice function in Example 2 satisfies (BTA) but not (SARP). It follows, therefore, that (SARP) and (BTA) are generally independent.) The choice function in Example 3 satisfies (R) without satisfying (RR), so that (R) does not imply (RR). This choice function does not satisfy (B), so that (R) does not imply (B). The choice function in Example 4 satisfies (B) but not (R), so that *a fortiori* it does not satisfy (RR). Thus (RR) is not implied by (B).

The choice function given by Example 5 satisfies (C) without satisfying (R), so that (C) does not necessarily imply (R). It can be seen that the choice function in Example 6 satisfies (FARP) but not (C), neither does it satisfy (WARP). Therefore, (FARP) does not imply (C) in general, neither does (FARP) imply (WARP). Finally we note that the choice function in Example 6 satisfies (SARP) but not (B), so that (SARP) does not imply (B) in general. Q. E. D.

Proof of Theorem 1. It has been shown by Blair *et al.* [5] that, if $n \geq 4$, there exists no CCR which satisfies (U), (BTA), (P), (I), (WD) and (PR) (see also Sen [23]). Our implication diagram shows that (FARP) is a stronger requirement on CCR than (BTA), so that *a fortiori* it is incompatible with (U), (P), (I), (WD) and (PR), establishing Theorem 1. Q. E. D.

Let R and S be, respectively, a binary relation on X and a subset of X. A sequence of relations $\{R_S^{(n)}\}_{n=1}^{\infty}$ is then defined recursively by $R_S^{(1)} = R$, $R_S^{(n)} = RR_S^{(n-1)} = \{(x, y) : (x, z) \in R \text{ and } (z, y) \in R_S^{(n-1)} \text{ for some } z \in S\}$ $(n \geq 2)$. The *transitive closure of R relative to S* is a binary relation which is defined by $T(R|S) = \cup_{n=1}^{\infty} R_S^{(n)}$. For simplicity we let $T(R) = T(R|X)$.

Proof of Theorem 2. For any profile $(R_1, \ldots, R_n)$, let $N(xR_iy)$ be the number of individuals who regard x to be at least as good as y. We define an R by $[xRy \Leftrightarrow N(xR_iy) \geq N(yR_ix)]$ for all $x, y \in X$ and define a CCR F by associating a choice function $C(S) = G_{T(R|S)}(S)$ with $(R_1, \ldots, R_n)$. It is easy to verify that this CCR satisfies (U), (B), (P), (I), (WD) and (PR) (see Bordes [6]). Our implication diagram shows that (SARP) is a weaker requirement on CCR than (B), so that *a fortiori* it is compatible with the requirements (U), (P), (I), (WD) and (PR). Hence Theorem 2. Q. E. D.

Proof of Theorem 3. (a) *Necessity proof.* Suppose that R has an extended ordering R^*. Let t be any finite positive integer and suppose that we have $x^1Px^2R \ldots Rx^t$ for some $x^1, x^2, \ldots, x^t$ in X. Then we have $x^1P^*x^2R^* \ldots R^*x^t$, which yields $x^1P^*x^t$, thanks to the transitivity of R^*. Thus we have (*not* $x^tR^*x^1$), which implies (*not* x^tRx^1). It follows that if R has an extended ordering, it has to be consistent.

(b) *Sufficiency proof*. Let the identity Δ be defined by $\Delta = \{(x, x) : x \in X\}$. We define a binary relation Q by

$$Q = \Delta \cup T(R). \tag{4}$$

We show that Q is a quasi-ordering. Reflexivity is obvious. In order to show its transitivity, let $(x, y), (y, z) \in Q$. If $(x, y), (y, z) \in T(R)$, we have $(x, z) \in T(R) \subset Q$. If $(x, y) \in \Delta$ (resp. $(y, z) \in \Delta$), we have $x = y$ (resp. $y = z$), so that $(x, z) \in Q$ follows from $(y, z) \in Q$ (resp. $(x, y) \in Q$). Q being a quasi-ordering, it has an extension that is an ordering (see, for example, Fishburn [10, Lemma 15.4]). If we can show that Q is an extension of R, we are home. For that purpose, we have to show that R is included in Q and P in P_Q (asymmetric component of Q). The former is obvious. To prove the latter, assume $(x, y) \in P$, which means $(x, y) \in R$ and $(y, x) \notin R$. From $(x, y) \in R$ it follows that $(x,y) \in Q$, so that we have only to prove that $(y,x) \notin Q$. Assume, therefore, that $(y,x) \in Q$. Clearly $(y,x) \notin \Delta$, otherwise we cannot have $(x,y) \in P$. It follows that $(y,x) \in T(R)$. When $(x,y) \in P$ is added to this, we obtain a PR-cycle, and this contradiction proves the theorem. Q. E. D.

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Chapter 11
Paretian Welfare Judgements and Bergsonian Social Choice*

In the wake of a harsh ordinalist criticism in the 1930s against the epistemological basis of the 'old' welfare economics created by Pigou, several attempts were made to salvage the wreckage of Pigou's research agenda by reformulating welfare economics altogether on the basis of *ordinal* and *interpersonally non-comparable* welfare information and nothing else. The seminal concept of the *Pareto principle* to the effect that a change from a state x to another state y can be construed as socially good if at least one individual is made better off without making anybody else worse off in return came to the fore, and the characterisation and implementation of the Pareto efficient resource allocations became the central exercise in the "new" welfare economics. However, since almost every economic policy cannot but favour some individuals at the cost of disfavouring others, there will be almost no situation of real importance where the Pareto principle can claim direct relevance.

Two distinct approaches were explored to rectify this unsatisfactory state of welfare economics. The first approach was the introduction of *compensation criteria* by Kaldor [12], Hicks [11], Scitovsky [19] and Samuelson [18], which endeavoured to expand the applicability of the Pareto principle by introducing hypothetical compensation payments between gainers and losers. According to Graaff [8, pp.84-85], "[t]he compensation tests all spring from a desire to see what can be said about social welfare or 'real national income'... without making interpersonal comparisons of well-being ... . They have com-

*First published in *Economic Journal*, Vol.109, 1999, pp.204-220. Reprinted in Wood, J. C. and M. McLure, eds., *Paul A. Samuelson: Critical Assessments of Contemporary Economists*, 2nd Series, London: Routledge, 2004, pp.378-396. This chapter capitalises on Suzumura [29], and generalises the main theorem that first appeared there. The basic idea was presented at the Fourth Osnabrück Seminar on Individual Decisions and Social Choice held at the University of Osnabrück, September 14, 1996, the Research Seminar at the Institute for Social and Economic Research held at Osaka University, January 22, 1997, the Far Eastern Meeting of the Econometric Society held at the Chinese University of Hong Kong, July 24-6, 1997, and the Fourth International Meeting of the Society for Social Choice and Welfare held at the University of British Colombia, July 3-6, 1998. I am grateful to Professors Nick Baigent, David Donaldson, Wulf Gaertner, Philippe Mongin, Prasanta Pattanaik, Koichi Tadenuma, Yongsheng Xu, and Akira Yamazaki for their helpful comments and discussions at several stages of the evolution of this chapter. Last but surely not least, my sincere thanks go to the editor in charge, Professor Timothy Besley, and the two referees of *Economic Journal* for their helpful comments and suggestions. Needless to say, I am solely, responsible for any remaining defects of this chapter.

mon origin in Pareto's definition of an increase in social welfare — that at least one man must be better off and no one worse off — but they are extended to situations in which some people are made worse off." The second attempt was the introduction of the novel concept of a *social welfare function* by Bergson [3] and Samuelson [17, Chapter 8], which is rooted in the belief that the analysis of the logical consequences of any value judgements, irrespective of whose ethical beliefs they represent, whether or not they are widely shared in the society, or how they are generated in the first place, is a legitimate task of welfare economics. The social welfare function is nothing other than the formal way of characterising such an ethical belief which is *rational* in the sense of being complete as well as transitive over the alternative states of affairs. A *Paretian* (or *individualistic*) social welfare function is one which judges in concordance with the Pareto principle if the latter does have relevance. It was Arrow [1, p. 108] who neatly crystallised the gist of this approach as follows: "[T]he 'new welfare economics' says nothing about choices among Pareto-optimal alternatives. The purpose of the social welfare function was precisely to extend the unanimity quasi-ordering to a full social ordering."

Capitalising on Graaff's and Arrow's insightful observations on the nature and significance of these two schools of thought, where the concept of an *extension* of the Pareto quasi-ordering plays a crucial role in common, this chapter examines the logical performance of the new welfare economics. To be more precise, we synthesise the two approaches to the new welfare economics and identify a condition under which the new welfare economics is logically impeccable. The usefulness of our condition is two-fold. In the first place, we may thereby check whether or not the hypothetical compensation criteria proposed by Kaldor, Hicks, Scitovsky and Samuelson can serve as a useful preliminary step towards final rational social choice. In the second place, we can thereby cast a new light on some recent attempts to define several plausible quasi-orderings on social welfare including Suppes [23], Sen [20; 21], Fine [7], Blackorby and Donaldson [4] and Madden [15] vis-à-vis the analytical scenario of the new welfare economics.

It is hoped that the preceding attempts in the new welfare economics along various routes can be systematically understood and neatly evaluated with reference to our analysis in this chapter.

1 Motivation and Illustration

To motivate our analysis intuitively, and to illustrate the nature of our central theorem neatly, let us begin by examining a situation where a policy-maker should make democratic collective decisions by paying due respect for opinions expressed by citizens. For simplicity, suppose that there are only two citizens, say 1 and 2, and only three options, say, x, y and z. Citizens are free to express their preference orderings on $X = \{x, y, z\}$. Without loss of generality, assume that citizens can express strict preferences only. Then there are six logically possible preference orderings, say α, β, γ, δ, ϵ and ζ on X, which may possibly be expressed.

Figure 1 describes these possible preference orderings, where options are arranged vertically with the more preferred options being located above the less preferred ones.

For example, α is one of the preference orderings, according to which x is preferred to y, y is preferred to z, hence x is preferred to z. Arranging these possible preference orderings vertically and horizontally, we may construct a box in Figure 2 with 36 cells, where the vertical (resp. horizontal) list refers to citizen 1's (resp. citizen 2's) expressed preference orderings. Clearly, each and every cell in Figure 2 represents a profile of the two citizens' expressed preference orderings. For example, the cell (α, β) represents a profile in which citizen 1 expresses α and citizen 2 expresses β.

What we call a *Bergson-Samuelson social welfare ordering* is nothing other than a preference ordering from the society's viewpoint specified for each and every cell in Figure 2 which guides the rational collective decision to be made by the policy-maker. A Bergson-Samuelson social welfare ordering is *Paretian* or *individualistic* if it accepts everything that the Pareto principle tells us about the social desirability of one option vis-à-vis the other. For simplicity, we will refer to "what the Pareto principle tells us about the social desirability of one option vis-à-vis the other" collectively as the *Paretian welfare judgements*. To be democratic, it is minimally required that the policy-maker is ready to accept the Paretian welfare judgements.

α	β	γ	δ	ϵ	ζ
x	x	y	y	z	z
y	z	x	z	x	y
z	y	z	x	y	x

Figure 1. *Possible Preference Orderings*

To what extent do the Paretian welfare judgements restrict the admissible class of Paretian Bergson-Samuelson social welfare orderings to be filled in each and every cell? Clearly the answer hinges squarely on the extent to which the two citizens agree on their individual preference orderings. Note that there are six cells (α, α), (β, β), (γ, γ), (δ, δ), (ϵ, ϵ) and (ζ, ζ), along the main diagonal in Figure 2, where the two citizens express exactly the same preference orderings α, β, γ, δ, ϵ and ζ, respectively. Therefore, by virtue of the Pareto principle and nothing else, the Paretian Bergson-Samuelson social welfare ordering corresponding to these cells cannot but be α, β, γ, δ, ϵ and ζ, respectively, so that nothing is left for any other principle to bridge the Paretian welfare judgements to the Paretian Bergson-Samuelson social welfare ordering. In all other cells in Figure 2, the extent of interpersonal agreements of preferences among citizens is less than perfect. However, there are two distinct categories to be identified and separately addressed. The first category consists of those cells where only one pair of options is left to be socially ordered in order to bridge the Paretian welfare judgements to the Paretian Bergson-Samuelson social welfare ordering, whereas the second category consists of those cells where at least two pairs of options are left to be socially ordered before the Paretian welfare judgements can be completed into the Paretian Bergson-Samuelson social welfare ordering.

	α	β	γ	δ	ϵ	ζ
α	α	xPy xPz	yPz xPz	yPz	xPy	
β	xPy xPz	β	xPz		zPy xPy	zPy
γ	yPz xPz	xPz	γ	yPx yPz		yPx
δ	yPz		yPz yPx	δ	zPx	yPx zPx
ϵ	xPy	zPy xPy		zPy	ϵ	zPy zPx
ζ		zPy	yPx	yPx zPx	zPy zPx	ζ

Figure 2. *Implications of the Pareto Principle*

Take, for example, the cell (α, β), where the Pareto principle is enough to tell us that xPy and xPz, that is to say, the policy-maker must accept that x is socially preferred to both y and z. Only one pair of options, viz. $\{y, z\}$, is left to be socially ordered before we arrive at a fully-fledged Paretian Bergson-Samueslon social welfare ordering, so that this cell is of the first category. If y (resp. z) is judged somehow socially better than z (resp. y) in this cell, then we have xPy, yPz and xPz, (resp. xPz, zPy and xPy), so that the Paretian Bergson-Samueslon social welfare ordering to be filled in this cell is uniquely determined to be α (resp. β). It is clear that the cells in Figure 2 which compose the first category consist of (α, β), (α, γ), (β, α), (β, ϵ), (γ, α), (γ, δ), (δ, γ), (δ, ζ), (ϵ, β),

(ϵ, ζ), (ζ, δ) and (ζ, ϵ). By definition, the step of deleting the residual indeterminacy in the cells of the first category is in fact nothing other than the *final* step rather than the *preliminary* step in the transition from the Paretian welfare judgements to the fully-fledged Paretian Bergson-Samuelson social welfare ordering. Since the asserted raison d'être of the new welfare economics lies in its preliminary role in expanding the Pareto principle in the presence of conflict of judgements among citizens without using anything that goes beyond intrapersonally ordinal and interpersonally non-comparable information on well-being, we will focus mostly on the cells (profiles) of the second category.[1]

Consider the cell (α, δ) which is clearly of the second category. Since the Pareto principle tells us that yPz and nothing else in this cell, the feasible candidates for the Paretian Bergson-Samuelson social welfare ordering in this cell are limited to α, γ and δ. Without loss of generality, suppose that α is the relevant social welfare ordering in this cell, and consider the set of partial welfare judgements which not only strictly extend the Paretian welfare judgements, but also are strictly subsumed in the Paretian Bergson-Samuelson social welfare ordering α. It is easy to check that two partial welfare judgements exist, which satisfy these two requirements, viz. (i) Q^1, which says that y is socially better than z and x is socially better than z, and (ii) Q^2, which says that x is socially better than y and y is socially better than z. Note that, according to Q^1, the policy-maker should not choose z, as it is dominated by x as well as by y in terms of social welfare. Likewise, according to Q^2, the policy-maker should choose neither y nor z, as they are both dominated by x in terms of social welfare. It then follows that there is only one option in the sex $X = \{x, y, z\}$ of all alternatives, viz. x, which is not excluded by any one of the two partial welfare judgements Q^1 and Q^2. As a matter of fact, this social choice of x is in full concordance with the choice according to the optimisation of the Paretian Bergson-Samuelson social welfare ordering α.

This remarkable result is not an accidental outcome of our fortuitous choice of a cell of the second category, viz. (α, δ), neither is it due to our cunning choice of a particular Paretian Bergson-Samuelson social welfare ordering α. Indeed, what our main theorem guarantees is that this property is in fact a robust result which holds for all profiles of citizens' preference orderings of the second category in a society with arbitrary number of citizens and options.

It is also important to realise the implication of this result. It suggests unambiguously that the research programme of the new welfare economics is logically impeccable in the following sense: For each and every Paretian Bergson-Samuelson social welfare ordering R, the social choice set from any opportunity set S in accordance with the optimisation of this R over S can be recovered by finding the undominated subsets of S for each and every partial preference relation, which is a *strict sub-relation* of R as well as a *strict extension* of the Paretian welfare judgements, and taking the intersection of these undominated subsets. Thus, the preliminary step advocated by the new welfare economics serves us

[1] The proportion occupied by cells of the second category over the total number of cells is $18/36 = 0.5$ in the situation with two citizens and three options, where citizens are allowed to express strict preference orderings only. It is easy, if tedious, to verify that this proportion increases to $120/169 \approx 0.71$ when two citizens are allowed to express occasional indifference among three options. For societies with more than two citizens and more than three options, this proportion will surely increase even further.

well in locating the rational choice exactly in terms of the Paretian Bergson-Samuelson social welfare ordering. In other words, the new welfare economics can indeed help the policy-maker to identify the rational social choice, at least in principle.

So much for the motivation and implication of our subsequent analysis. Let us now proceed to the proper analysis in a general setting.

2 Maximal Set, Greatest Set and Extensions of a Binary Relation

To facilitate our analysis for a society with arbitrary number of individuals and alternatives, let us summarise some basic properties of binary relations, maximal sets and greatest sets and present an extension theorem for consistent binary relations, which will play a crucial role in what follows.

Let X be the universal set of alternatives. A *binary relation* on X is a subset R of $X \times X$. It is customary to write, for all $x, y \in X$, xRy if and only if $(x, y) \in R$. When a binary relation R satisfies *completeness* (*For all* $x, y \in X, x \neq y$ *implies* $\{(x, y) \in R$ or $(y, x) \in R\}$), *reflexivity* (*For all* $x \in X, (x, x) \in R$) and *transitivity* (*For all* $x, y, z \in X, \{(x, y) \in R$ & $(y, z) \in R\}$ *implies* $(x, z) \in R$), we say that R is an *ordering* on X. If R satisfies reflexivity and transitivity, but not necessarily completeness, we say that R is a *quasi-ordering*.

For any binary relation R on X, let $P(R)$ and $I(R)$ denote, respectively, the *asymmetric part* of R and the *symmetric part* of R, which are defined by $P(R) = \{(x, y) \in X \times X \mid (x, y) \in R \;\&\; (y, x) \notin R\}$ and $I(R) = \{(x, y) \in X \times X \mid (x, y) \in R \;\&\; (y, x) \in R\}$. If R stands for an agent's preference relation over the set X of options, where xRy for some $x, y \in X$ means that this agent judges x to be at least as good as y, $P(R)$ and $I(R)$ stand, respectively, for his/her *strict preference relation* and *indifference relation*. If R is transitive, $P(R)$ as well as $I(R)$ satisfies transitivity.

For any binary relation R and any non-empty subset S of X, an element $x \in S$ is an *R-maximal element* of S if $(y, x) \notin P(R)$ holds for all $y \in S$. The set of all R-maximal elements of S is the *R-maximal set* of S, to be denoted by $M(S, R)$. Likewise, an element $x \in S$ is an *R-greatest element* of S if $(x, y) \in R$ holds for all $y \in S$. The set of all R-greatest elements of S is the R-greatest set of S, to be denoted by $G(S, R)$. The best known example of the R-maximal set $M(S, R)$ may be the set of Pareto efficient allocations, where S stands for the set of all feasible allocations and R stands for the *unanimity quasi-ordering* such that $(x, y) \in R$ holds if and only if everybody in the society regards x to be at least as good as y. The best known example of the R-greatest set $G(S, R)$ may be the demand correspondence of a consumer, where S denotes his/her budget set and R denotes his/her preference ordering.

The following lemma, which is a straightforward consequence of the definitions of a maximal set and a greatest set, will prove useful in our analysis. A formal proof of this lemma is available in Sen [20, Chapter 1*] if necessary. See also Sen [22] and Suzumura [27, Chapter 2].

Lemma 1

(a) $G(S,R) \subset M(S,R)$ *holds for all* (S,R).
(b) $G(S,R) = M(S,R)$ *holds for all* (S,R) *such that* R *is complete*.

A *choice function* C on a family K of non-empty subsets of X maps each and every $S \in K$ into a non-empty subset $C(S)$ of S, which is called the *choice set* from the *opportunity set* S. A choice function C on K is *rational* if and only if there exists an underlying preference ordering R, to be called the *rationalisation* of C, such that $C(S) = G(S,R)$ holds for all $S \in K$. Thus, a *rational choice* is a choice in accordance with the optimisation of an underlying preference ordering.[2].

For any binary relation R on X, a binary relation R^* on X is called an *extension* of R if and only if $R \subset R^*$ and $P(R) \subset P(R^*)$. Thus, an extension R^* of R retains all the information that R already contains, and goes possibly further. Note, in particular, that the asymmetric part $P(R)$ of R must be subsumed in the asymmetric part of R^*. When R^* is an extension of R, R is in turn called a *sub-relation* of R^*.

Let $\Sigma(R)$ denote the set of all sub-relations of R. Then we have the following simple properties of the maximal sets and the greatest sets:

Lemma 2

If $Q \in \Sigma(R)$, *then* $M(S,R) \subset M(S,Q)$ *and* $G(S,Q) \subset G(S,R)$.

Proof: Obvious from the respective definitions of a maximal set, a greatest set and an extension of a binary relation. ‖

Under what conditions can there be an extension of a binary relation that satisfies the axioms of an ordering? As an auxiliary step in answering this crucial question of the existence of an *ordering extension*, we introduce the following two weaker versions of the transitivity axiom.

First, a binary relation R is *acyclic* if and only if there exists no finite subset $\{x^1, x^2, \dots, x^t\}$ of X, where $2 \leq t < +\infty$, such that $(x^1, x^2) \in P(R), (x^2, x^3) \in P(R), \dots, (x^t, x^1) \in P(R)$. Second, a binary relation R is *consistent* if and only if there exists no finite subset $\{x^1, x^2, \dots, x^t\}$ of X, where $2 \leq t < +\infty$, such that $(x^1, x^2) \in P(R), (x^2, x^3) \in R, \dots, (x^t, x^1) \in R$. It is clear that transitivity of R implies consistency thereof, while consistency of R implies acyclicity thereof. The converse of each one of these implications is not true in general. Indeed, if we define R^1 and R^2 on $X = \{x, y, z\}$ by

$$R^1 = \{(x,y),(y,z),(z,y)\}; R^2 = \{(x,y),(y,z),(z,y),(x,z),(z,x)\},$$

[2]In the general theory of rational choice functions, a choice function is called *rational* if an underlying preference relation exists irrespective of whether the preference relation in question satisfies the axioms of an ordering. A rational choice function whose underlying preference relation satisfies the axioms of an ordering is called a *fully rational* choice function. See, for example, Suzumura [24; 27, Chapter 2]. Since we are concerned in this chapter only with a choice function which is rationalised by a Paretian Bergson-Samuelson social welfare ordering, however, we can do without introducing the concept of *degrees* of rationality altogether.

it is clear that R^1 is consistent but not transitive, whereas R^2 is acyclic but not consistent. If R is complete, however, R is consistent if and only if it is transitive.

We are now ready to state the following basic extension theorem.

Lemma 3 (Suzumura [24, Theorem 3; 27, Theorem A(5)][3]

A binary relation R has an ordering extension if and only if it is consistent.

Let $\Omega(X)$ stand for the set of all reflexive and consistent binary relations on X. It is clear that $\Sigma(R) \subset \Omega(X)$ holds for any ordering R on X by virtue of Lemma 3. It is also clear that $R \in \Sigma(R)$ holds for any binary relation R, viz. any binary relation R is an extension of itself. Noting this fact, we call a binary relation $Q \in \Sigma(R)\backslash\{R\}$ a *strict sub-relation* of R. R is then called a *strict extension* of Q.

So much for purely technical preliminaries. Let us now formally introduce our social choice problem.

3 The Pareto Quasi-Ordering and Bergson-Samuelson Social Welfare Orderings

Let $X := \{x, y, z, \ldots\}$ and $N := \{1, 2, \ldots, n\}$ be the set of all social states and the set of all individuals in the society, where $3 \leq \#X$ and $2 \leq n := \#N < +\infty$.[4] A *social state* means a complete description of economic, social and all other features of the world which may possibly influence the well-being of individuals.

Each individual $i \in N$ is assumed to have a weak preference ("at least as good as") relation R_i on X, which satisfies the axioms of an ordering on X, such that xR_iy holds if and only if x is judged by i to be at least as good as y. By definition, $P(R_i)$ and $I(R_i)$ stand for i's *strict preference relation* and his/her *indifference relation*, respectively.

Given a list of individual preference ordering $\boldsymbol{R}^N = (R_1, R_2, \ldots, R_n)$, to be called a *profile* for short, we define the *Pareto quasi-ordering* $\rho(\boldsymbol{R}^N)$ by

$$\rho(\boldsymbol{R}^N) = \cap R_i \text{ over all } i \in N.^5 \tag{1}$$

By definition, $(x, y) \in \rho(\boldsymbol{R}^N)$ if and only if xR_iy for all $i \in N$, whereas $(x, y) \in P(\rho(\boldsymbol{R}^N))$ if and only if xR_iy for all $i \in N$, and $xP(R_i)y$ for at least one $i \in N$. Within this conceptual framework, the problem confronted by the two schools of the new welfare economics may be neatly formulated as follows.

Recollect that the compensation criteria were designed to extend the applicability of the Pareto principle through hypothetical compensatory payments between gainers

[3] Since transitivity is a sufficient but not necessary condition for consistency, this theorem is in fact a generalisation of the classical theorem to Szpilrajn [30]. See also Arrow [1, p.64] and Sen [20, Chapter 1*] for the role and importance of the extension theorems.

[4] For each set A, $\#A$ stands for the number of elements in A.

[5] R_i being an ordering for every $i \in N$, $\rho(\boldsymbol{R}^N)$ satisfies reflexivity and transitivity, which is why $\rho(\boldsymbol{R}^N)$ is called the Pareto quasi-ordering.

and losers. Let Q denote the generic binary relation representing the partial welfare judgements thus defined. The first task for this school of thought is to ensure that Q is a strict extension of $\rho(\boldsymbol{R}^N)$, viz. $\rho(\boldsymbol{R}^N) \in \Sigma(Q)\backslash\{Q\}$, since this school intends to go beyond the Pareto principle without losing what the latter principle already informs us of social welfare. But this is only half of the full story. Since the compensation criteria provide only a preliminary step towards final social choice, which should be rationalised by a Paretian Bergson-Samuelson social welfare ordering, the mission will be left unaccomplished if what is meant to be a preliminary step turns out to preclude the possibility of final rational social choice. In other words, for the success of the compensationist school of thought, it is necessary that Q should be a strict sub-relation of at least one Paretian Bergson-Samuelson social welfare ordering R so that we have $Q \in \Sigma(R)\backslash\{R\}$.

We have thus identified the research programme of the new welfare economics as that of defining a principle of compensation between gainers and losers so as to generate partial welfare judgements Q such that (i) $\rho(\boldsymbol{R}^N) \in \Sigma(Q)\backslash\{Q\}$, and (ii) $Q \in \Sigma(R)\backslash\{R\}$ for some Paretian Bergson-Samuelson social welfare ordering R. It follows from this observation and Lemma 3 that this research programme will be vacuous unless Q is guaranteed to be consistent. This is a useful remark, as it enables us to check whether the promise of the compensationist new welfare economics can be logically fulfilled.

4 The Recoverability Theorem

The research programme of the new welfare economics can be located in a wider perspective with fruitful implications. Let R be a Paretian Bergson-Samuelson social welfare ordering corresponding to a given profile $\boldsymbol{R}^N = (R_1, R_2, \ldots, R_n)$ of individual preference orderings. Let $\Theta(\boldsymbol{R}^N, R)$ stand for the set of all partial welfare judgements, to be called the *test relations* for short,[6] which are *strict* extension of $\rho(\boldsymbol{R}^N)$ as well as *strict* sub-relation of R:

$$\Theta(\boldsymbol{R}^N, R) := \{Q \subset X \times X \mid \rho(\boldsymbol{R}^N) \in \Sigma(Q)\backslash\{Q\} \ \& \ Q \in \Sigma(R)\backslash\{R\}\}. \tag{2}$$

It is worthwhile to repeat why we should exclude $\rho(\boldsymbol{R}^N)$ and R from the set $\Theta(\boldsymbol{R}^N, R)$. The reason is squarely rooted in the very nature of the new welfare economics. Recollect that the research programme of this school of welfare economics is to go beyond the Pareto principle and to provide a preliminary step towards the Paretian Bergson-Samuelson social welfare ordering that rationalises the final social choice. Thus, $\rho(\boldsymbol{R}^N)$ as well as R cannot possibly qualify as the test relations in our analytical scenario, as $\rho(\boldsymbol{R}^N)$ (resp. R) does not in fact go beyond the Pareto quasi-ordering (resp. is not in fact a preliminary step in locating R).

At this juncture of our analysis, it is useful to observe that Lemma 1 and Lemma 2 may assert that

$$G(S, R) = M(S, R) \subset M(S, Q) \tag{3}$$

[6] This convenient terminology was suggested to me by one of the referees of *Economic Journal*.

holds for any $S \in K$ and $Q \in \Theta(\boldsymbol{R}^N, R)$, So that we may assert that

$$\forall S \in K : C(S) := G(S, R) \subset \cap M(S, Q) \textit{ over all } Q \in \Theta(\boldsymbol{R}^N, R), \tag{4}$$

holds, where C is the *social choice function* on K, which is rationalised by the Paretian Bergson-Samuelson social welfare ordering R. Thus, by defining the intersection of all the maximal sets with respect to each and every test relation $Q \in \Theta(\boldsymbol{R}^N, R)$, we can locate the area from which the final rational social choice in accordance with the Paretian Bergson-Samuelson social welfare ordering cannot escape.

Going one step further, suppose that (4) can be strengthened into the following set-theoretic equality:

$$\forall S \in K : C(S) := G(S, R) = \cap M(S, Q) \textit{ over all } Q \in \Theta(\boldsymbol{R}^N, R). \tag{5}$$

According to (5), *the social choice function C, which is rationalisable by the Paretian Bergson-Samuelson social welfare ordering R, can be exactly recovered by the maximisation of each and every test relation* $Q \in \Theta(\boldsymbol{R}^N, R)$. Therefore, it makes sense to assert that the search for the test relations that are strict extensions of the Pareto quasi-ordering $\rho(\boldsymbol{R}^N)$ as well as strict sub-relations of the Paretian Bergson-Samuelson social welfare ordering R is a legitimate and effective preliminary step for final rational social choice if and only if (5) holds true. It is in this sense that the search for the conditions under which (5) holds true is of crucial importance for the logical completeness of the new welfare economies.

To orient our analysis, consider the following example:

Example 1: Suppose that $X = \{x, y, z\}$ and $\boldsymbol{R}^N = (R_1, R_2)$ are such that $R_1 = \mathit{\Delta}(X) \cup \{(x, y), (y, z), (x, z)\}$ and $R_2 = \mathit{\Delta}(X) \cup \{(y, x), (x, z), (y, z)\}$, where $\mathit{\Delta}(X) := \{(x, x), (y, y), (z, z)\}$, denotes the *diagonal binary relation* on X. Let the Paretian Bergson-Samuelson social welfare ordering be given by $R = \mathit{\Delta}(X) \cup \{(x, y), (y, z), (x, z)\}$. In this situation, we have $\rho(\boldsymbol{R}^N) = \mathit{\Delta}(X) \cup \{(y, z), (x, z)\}$ and there exists no binary relation Q that satisfies $\rho(\boldsymbol{R}^N) \in \Sigma(Q)\backslash\{Q\}$ and $Q \in \Sigma(R)\backslash\{R\}$. Thus, we have $\Theta(\boldsymbol{R}^N, R) = \varnothing$ for this $\boldsymbol{R}^N$ and R. $\|$

The message of this example is simple. If the degree of interpersonal difference of preferences is small enough as in Example 1, it may turn out that $\Theta(\boldsymbol{R}^N, R)$ is empty for the given profile $\boldsymbol{R}^N$ of individual preference orderings and the given Paretian Bergson-Samuelson social preference ordering R. It is to confine ourselves to the interesting situations that we introduce the following assumption of *non-triviality*:

Assumption NT
$\Theta(\boldsymbol{R}^N, R)$ is non-empty for the given profile $\boldsymbol{R}^N = (R_1, R_2, \ldots, R_n)$ of individual preference orderings and the given Paretian Bergson-Samuelson social welfare ordering R.

Mild and innocuous though the assumption NT may look, it enables us to prove the following recoverability theorem:

Recoverability Theorem

For any given profile $\boldsymbol{R}^N = (R_1, R_2, \ldots, R_n)$ of individual preference orderings, the rational social choice in accordance with the specified Paretian Bergson-Samuelson social welfare ordering R can be fully recovered by finding the maximal set for each and every test relation $Q \in \Theta(\boldsymbol{R}^N, R)$ and taking the intersection of these maximal sets if and only if the assumption NT *is satisfied.*

It should be clear that the Recoverability Theorem verifies the general validity of what we have illustrated in Section 2 in terms of a simple case of two citizens and three options, and for the profile of citizens' preference orderings of the second category. Clearly, it also suffices to establish the logical completeness of the research programme of the new welfare economics. Since the proof of the Recoverability Theorem is slightly involved, it will be relegated into Section 5, which may be neglected by those who are not interested in minor technicalities.

There may be some readers who are interested in knowing whether or not a stronger version of the recoverability property holds. Let $\Omega^*(X)$ be the set of all quasi-orderings on X, and define the set $\Theta^*(\boldsymbol{R}^N, R)$ by

$$\Theta^*(\boldsymbol{R}^N, R) := \Omega^*(X) \cap \Theta(\boldsymbol{R}^N, R). \tag{6}$$

By construction, $Q \in \Theta^*(\boldsymbol{R}^N, R)$ holds if and only if (i) Q is a quasi-ordering on X, (ii) Q is a strict extension of the Pareto quasi-ordering $\rho(\boldsymbol{R}^N)$, and (iii) Q is a strict sub-relation of R. Since it is clear that $\Theta^*(\boldsymbol{R}^N, R) \subset \Theta(\boldsymbol{R}^N, R)$ holds, we may assert that

$$\begin{aligned} \forall S \in K : C(S) := G(S, R) &\subset \cap M(S, R) \textit{ over all } Q \in \Theta(\boldsymbol{R}^N, R) \\ &\subset \cap M(S, R) \textit{ over all } Q \in \Theta^*(\boldsymbol{R}^N, R) \end{aligned} \tag{7}$$

holds. Thus, if we could prove that the recoverability of $C(S)$ in terms of $\Theta^*(\boldsymbol{R}^N, R)$ holds, viz.

$$\forall S \in K : C(S) := G(S, R) = \cap M(S, Q) \textit{ over all } Q \in \Theta^*(\boldsymbol{R}^N, R) \tag{8}$$

holds, it would follow from (7) that the recoverability of $C(S)$ in terms of $\Theta(\boldsymbol{R}^N, R)$ must hold *a fortiori*. However, the recoverability property (8) is not true in general, as we can easily check in terms of the Example 2 and Example 3 in Section 5. Plausible and desirable though it may look, the recoverability of the Bergsonian social choice in terms of the maximisation of each and every Paretian quasi-ordering is a target which is unattainable in general.[7]

[7]This is not to deny the possibility that some necessary and sufficient conditions can be identified for the recoverability property in terms of $\Theta^*(\boldsymbol{R}^N, R)$ to hold. For example, if the set of Pareto non-comparable pairs consists of only two pairs, we have succeeded in identifying the set of necessary and sufficient conditions for the recoverability property in terms of $\Theta^*(\boldsymbol{R}^N, R)$ to hold. Even in this simple case, however, the necessary and sufficient conditions in question are fairly complicated, and the proof of the recoverability property is rather involved.

5 Proof and Counter-Examples

5.1. Proof of the Recoverability Theorem

The 'only if' part being obviously true, we have only to prove the 'if' part by *reductio ad absurdum*. Suppose to the contrary that there is an $S \in K$ and an $x \in \cap M(S, Q)$ *over all* $Q \in \Theta(\boldsymbol{R}^N, R)$ such that $x \notin G(S, R)$, which means that

$$\exists z \in S : (z, x) \in P(R) \tag{9}$$

and

$$\forall Q \in \Theta(\boldsymbol{R}^N, R) : (z, x) \notin P(Q), \tag{10}$$

where use is made of the completeness of R. Thus:

$$\forall Q \in \Theta(\boldsymbol{R}^N, R) : Q \subset\subset R \textit{ and } P(Q) \subset\subset P(R). \tag{11}$$

There are two cases to be considered.

Case 1: $Q^0 \cup \{(z, x)\} \subset\subset R$ for *some* $Q^0 \in \Theta(\boldsymbol{R}^N, R)$.

Case 2: $Q \cup \{(z, x)\} = R$ for *all* $Q \in \Theta(\boldsymbol{R}^N, R)$.

It should be clear that Case 2 implies that $\Theta(\boldsymbol{R}^N, R)$ is in fact a singleton set, say $\{Q^*\}$, where $Q^* \cup \{(z, x)\} = R$. However, it follows from the definition of $\Theta(\boldsymbol{R}^N, R)$ and the assumption NT that there exist at least two distinct pairs, say $\{a, b\}$ and $\{u, v\}$, such that, if we denote the relation between a and b (resp. u and v) in accordance with R by $(a * b)$ (resp. $(u * v)$), $Q^1 = \rho(\boldsymbol{R}^N) \cup \{(a * b)\}$ and $Q^2 = \rho(\boldsymbol{R}^N) \cup \{(u * v)\}$ must both belong to $\Theta(\boldsymbol{R}^N, R)$, which is a contradiction.

Consider now Case 1 and define

$$Q^{**} = Q^0 \cup \{(z, x)\}. \tag{12}$$

Our task is to verify that $Q^{**} \in \Theta(\boldsymbol{R}^N, R)$. Since $Q^{**} \subset\subset R$ holds by definition, we have only to check that $\rho(\boldsymbol{R}^N) \in \Sigma(Q^{**}) \backslash \{Q^{**}\}$ and $Q^{**} \in \Sigma(R)$. Note that $\rho(\boldsymbol{R}^N) \in \Sigma(Q^0) \backslash \{Q^0\}$ and (12) imply that $\rho(\boldsymbol{R}^N) \subset Q^0 \subset\subset Q^{**}$. To show that $P(\rho(\boldsymbol{R}^N)) \subset P(Q^{**})$, observe that $(a, b) \in P(Q^{**})$ if and only if either

$$(a, b) = (z, x), (b, a) \notin Q^0 \textit{ and } (b, a) \neq (z, x) \tag{13}$$

or

$$(a, b) \in P(Q^0) \textit{ and } (b, a) \neq (z, x) \tag{14}$$

holds, where use is made of (12). Suppose $(a, b) \in P(\rho(\boldsymbol{R}^N))$. It follows from $\rho(\boldsymbol{R}^N) \in \Sigma(Q^0)$ that $(a, b) \in P(Q^0)$, which implies that $(b, a) \notin Q^0$. If it happens to be the case that $(a, b) = (z, x)$, then $(b, a) = (x, z) \neq (z, x)$ holds, since otherwise $(a, b) = (x, z) \in P(Q^0) \subset P(R)$ in contradiction with $(z, x) \in P(R)$. Thus (13) must be the case, hence $(a, b) \in P(Q^{**})$ obtains. On the other hand, if $(a, b) \neq (z, x)$, then we cannot have

$(b, a) = (z, x)$, since otherwise $(a, b) \in P(\rho(\boldsymbol{R}^N)) \subset P(Q^0)$ will have to imply $(x, z) \in P(Q^0) \subset P(R)$, a contradiction. Hence (14) must hold, so that $\rho(\boldsymbol{R}^N) \in \Sigma(Q^{**})\backslash\{Q^{**}\}$.

To verify that $Q^{**} \in \Sigma(R)$, note that $Q^0 \in \Sigma(R), (z, x) \in P(R)$ and (12) imply $Q^{**} \subset R$. If $(a, b) \in P(Q^{**})$, either (13) or (14) holds. If (13) is the case, (9) implies that $(a, b) \in P(R)$. On the other hand, (14) and $Q^0 \in \Sigma(R)$ imply $(a, b) \in P(Q^0) \subset P(R)$. Thus $Q^{**} \in \Sigma(R)$ must be true.

We have thus shown that $Q^{**} \in \Theta(\boldsymbol{R}^N, R)$. However, since $(z, x) \in P(Q^{**})$ by definition, this contradicts (10) for this Q^{**}. Therefore, we obtain

$$\forall S \in K : C(S) := G(S, R) \supset \cap M(S, Q) \textit{ over all } Q \in \Theta(\boldsymbol{R}^N, R), \tag{15}$$

which completes the proof of (5) in view of (4). ‖

5.2. Counter-Examples

Example 2: Let $S = X = \{x, y, z\}$, $N = \{1, 2\}$, and let a profile $\boldsymbol{R}^N = (R_1, R_2)$ of individual preference orderings be defined by $R_1 = \mathit{\Delta}(X) \cup \{(z, x), (x, y), (z, y)\}$ and $R_2 = \mathit{\Delta}(X) \cup \{(x, y), (y, z), (x, z)\}$. Clearly, the Pareto quasi-ordering is given in this case by $\rho(\boldsymbol{R}^N) = \mathit{\Delta}(X) \cup \{(x, y)\}$. If a Paretian Bergson-Samuelson social welfare ordering R is specified by $R = \mathit{\Delta}(X) \cup \{(z, x), (x, y), (z, y)\}$, $\Theta(\boldsymbol{R}^N, R)$ consists of two relations Q^1 and Q^2 such that $Q^1 = \rho(\boldsymbol{R}^N) \cup \{(z, y)\}$ and $Q^2 = \rho(\boldsymbol{R}^N) \cup \{(z, x)\}$. It follows that we have $G(S, R) = \{z\} = M(S, Q^1) \cap M(S, Q^2)$, vindicating the recoverability theorem in terms of $\Theta(\boldsymbol{R}^N, R)$. Note, however, that $\Theta^*(\boldsymbol{R}^N, R)$ consists only of Q^1, because Q^2 will have to be expanded so as to include $\{(z, y)\}$ in order for it to satisfy the axiom of transitivity. However, doing this cannot but imply that the expanded Q^2 must coincide with R. Then we have $G(S, R) = \{z\} \subset\subset \{x, z\} = M(S, Q^1)$, so that the recoverability property does not hold in terms of $\Theta^*(\boldsymbol{R}^N, R)$. ‖

Example 3: Let $S = X = \{x, y, z, w\}$, $N = \{1, 2\}$, and let a profile $\boldsymbol{R}^N = (R_1, R_2)$ of individual preference orderings be defined by $R_1 = \mathit{\Delta}(X) \cup \{(x, w), (w, y), (y, z), (x, y), (x, z), (w, z)\}$ and $R_2 = \mathit{\Delta}(X) \cup \{(w, x), (x, z), (z, y), (w, z), (w, y), (x, y)\}$. Clearly, the Pareto quasi-ordering is given in this case by $\rho(\boldsymbol{R}^N) = \mathit{\Delta}(X) \cup \{(x, y), (w, y), (w, z)\}$. If a Paretian Bergson-Samuelson social welfare ordering R is specified by $R = \mathit{\Delta}(X) \cup \{(w, z), (z, x), (x, y), (w, x), (w, y), (z, y)\}$, $\Theta^*(\boldsymbol{R}^N, R)$ consists of the following three relations: Q^1, Q^2 and Q^3 such that $Q^1 = \rho(\boldsymbol{R}^N) \cup \{(w, x)\}$ and $Q^2 = \rho(\boldsymbol{R}^N) \cup \{(z, y)\}$ and $Q^3 = \rho(\boldsymbol{R}^N) \cup \{(w, x), (z, y)\}$, so that $G(\{x, z\}, R) = \{z\} \subset\subset \{x, z\} = M(\{x, z\}, Q^1) \cap M(\{x, z\}, Q^2) \cap M(\{x, z\}, Q^3)$. Thus, the recoverability property does not hold in terms of $\Theta^*(\boldsymbol{R}^N, R)$. ‖

6 Concluding Remarks

To assert the logical impeccability of the research programme of the new welfare economics is one thing, and to assert the actual implementability of the logically impeccable programme is quite another. This chapter was mainly concerned with accomplishing the first task by establishing the recoverability of the Pareto-compatible Bergsonian social choice through the maximisation of the Pareto-inclusive test relations. However, we could identify *en route* a condition to be satisfied by the eligible Pareto-inclusive test relations. The condition in question is the logical requirement of consistency, which was first introduced by Suzumura [24; 27, pp. 8-11], and it enables us to check whether or not various test relations proposed in the literature are capable of implementing the research programme of the new welfare economics.

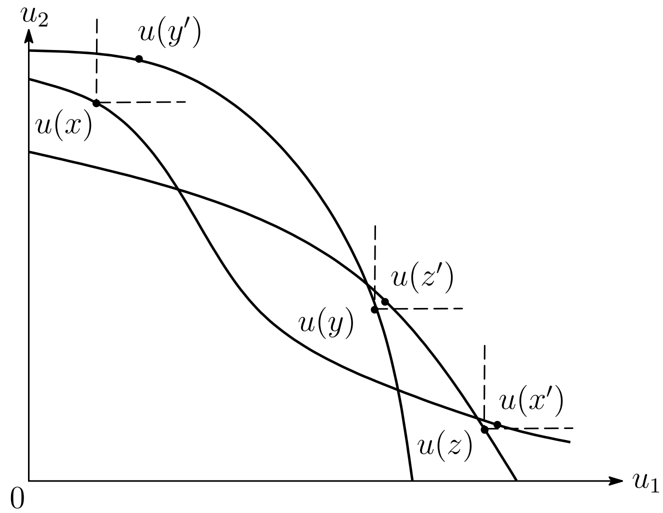

Figure 3. *Inconsistency of the Kaldor, Hicks and Scitovsky Compensation Principles*

Note: The curves depicted in this figure describe the utility possibility frontiers corresponding to various situations. Note that, according to each of the Kaldor, Hicks and Scitovsky principles, y is better than x, z is better than y, but x is better than z.

Take, for example, the hypothetical compensation principles proposed by Kaldor [12], Hicks [11], Scitovsky [19] and Samuelson [18]. The condition of consistency alone is sufficient to disqualify the Kaldor compensation principle, the Hicks compensation principle and the Scitovsky compensation principle as workable guideposts for final rational social choice. Although the failure of these principles is well vindicated by Arrow [1, Chapter IV], Chipman and Moore [6], Graaff [8, Chapters IV and V] and Suzumura [26; 28; 29], among others, the essence of their failure can be neatly illustrated in Figure 3.[8] The three

[8] The piecemeal welfare criteria à la Little [13; 14] and Mishan [16], which pay due attention to distributional equity considerations along with allocative efficiency considerations, can improve the logical score of the Kaldor-Hicks compensationist approach, but this progress is somewhat vacuous in the sense that the crucial value judgements on distributional equity are left completely unspecified. See Arrow [2, p.927], according to whom "[t]he hard problem ... arises at the point where Little and everyone else stops. It is all very well to say that the effects of a proposed change on income distribution must be taken into account in deciding on the desirability of the change: but how do we describe a distribution of real income? Admittedly, the choice between two income distributions is the result of a value judgement; but how do we even formulate such judgement?" See also Chipman and Moore [6] and Suzumura [26; 29], among others, for critical examination of this line of thought.

curves in this figure describe the utility possibility frontiers corresponding to three social states x, y and z. According to each of the Kaldor, Hicks and Scitovsky compensation principles, y is better than x, x is better than z, and z is better than y. Thus, these principles generate a test relation which is not acyclic, hence not consistent, so that these compensation principles are incapable of implementing the research programme of the new welfare economics.

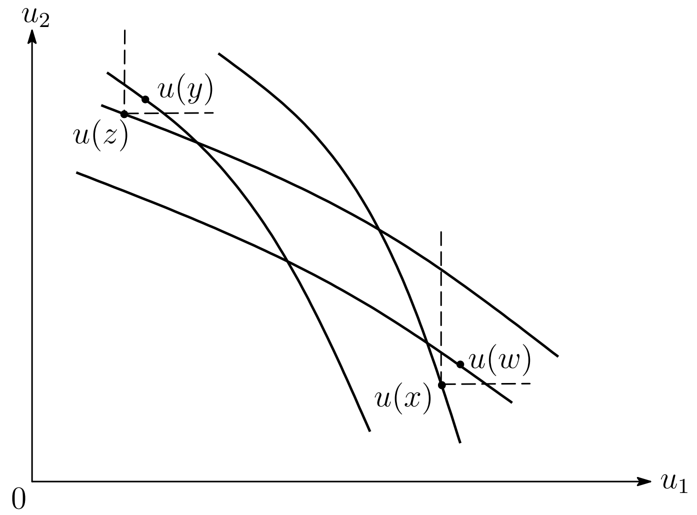

Figure 4. *Incompatibility between the Pareto Principle and the Samuelson Compensation Principle*

Note: The curves depicted in this figure describe the utility possibility frontiers corresponding to various situations. If the Samuelson compensation principle is an extension of the Pareto principle x is judged socially better than y, y is judged socially better than z, z is judged socially better than w, but w is judged socially better than x.

The verdict on the Samuelson compensation principle, which is defined in terms of a *uniform outward shift of the utility possibility frontiers*, is quite different. Indeed, the Samuelson principle can actually generate a test relation which is transitive, and hence consistent, but it fails to define a strict extension of the Pareto quasi-ordering in general.[9] This failure is illustrated in Figure 4, which shows that the Samuelson compensation principle may generate a test relation which cannot be compatible with the Pareto principle.

How about many interesting proposals in the more recent literature, which tried to construct some relevant Pareto-inclusive quasi-orderings? To cite just a few salient examples, the *grading principle of justice à la Suppes* [23] under the *axiom of identity* introduced by Sen [20, p.156] or the *principle of acceptance* due to Harsanyi [10, p.52], the *partial comparability approach* developed by Sen [21] and Fine [7], and some plausible

[9]See also a recent work by Gravel [9] which shows how restrictive is the class of economies which are free from this logical difficulty of the Samuelson compensation principle.

quasi-orderings introduced by Blackorby and Donaldson [4] immediately suggest themselves.[10] One may be led to think that an embarrassment of riches is a real possibility here, but a crucial problem still remains.[11] *All these proposed quasi-orderings are based on interpersonal welfare comparisons in one form or the other.* In this sense, they are not in fact in harmony with the informational basis of the new welfare economics.

It seems to us that defining the Pareto-inclusive test relations which can pass the crucial test of consistency without requiring anything beyond the informational basis characterised by *ordinalism* and *interpersonal non-comparability* is by no means an easy task.[12] This observation should not surprise anybody, however, as "nothing of much interest can be said on justice without bringing in some interpersonal comparability (Sen [20, p.150]."

[10]See also Madden [15].

[11]It should also be added that the recoverability of the Bergsonian social choice in terms of the maximisation of the Pareto-inclusive quasi-orderings does not hold in general.

[12]In the same spirit, one referee pose an interesting question, which reads as follows: "it could be of interest to look at whether there are any criteria which utilise information similar to that used by compensation criteria (information about positions being compared and situations that can be reached through redistribution) that are both anonymous and satisfy the Pareto criterion. A negative answer would put the final nail into the new welfare economics approach." This chapter stops short of putting the final nail into the new welfare economics, but this suggestion is surely interesting to explore.

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Chapter 12
Arrovian Aggregation in Economic Environments: How Much Should We Know about Indifference Surfaces?*

1 Introduction

From Arrow's celebrated theorem of social choice, it is well known that the aggregation of individual preferences into a social ordering cannot make the social ranking of any pair of alternatives depend only on individual preferences over that pair (this is the famous axiom of Independence of Irrelevant Alternatives). Or, more precisely, it cannot do so without trespassing basic requirements of unanimity (the Pareto principle) and anonymity (even in the very weak version of non-dictatorship). This raises the following question: What additional information about preferences would be needed in order to make aggregation of preferences possible, and compatible with the basic requirements of unanimity and anonymity?

In the last decades, the literature on social choice has explored several paths and gave interesting answers to this question. The main avenue of research has been, after Sen [18] and d'Aspremont and Gevers [7], the introduction of information about utilities, and it has been shown that the classical social welfare functions, and less classical ones, could be obtained with the Arrovian axiomatic method by letting the social preferences take account of specific kinds of utility information with interpersonal comparability.

In this chapter, we focus on the introduction of additional information about preferences that is not of the utility sort. In other words, we retain a framework with purely

*First published in *Journal of Economic Theory*, Vol.124, 2005, pp.22-44 as a joint paper with M. Fleurbaey and K. Tadenuma. We thank A. Leroux for a stimulating discussion, A. Trannoy and anonymous referees of *Journal of Economic Theory* for helpful comments, and participants at seminars in Cergy, Rochester, Waseda and Hitotsubashi, and the 5th International Meeting of the Society for Social Choice and Welfare in Alicante. Financial supports from the Ministry of Education, Culture, Sports, Science and Technology of Japan through Grant-in-Aid for Scientific Research No.10045010 ("Economic Institutions and Social Norms: Evolution and Transformation") and the 21st Century Center of Excellence Project on the Normative Evaluation and Social Choice of Contemporary Economic Systems are gratefully acknowledged.

ordinal and interpersonally non-comparable preferences. The kind of additional information that we study is about the shapes of indifference surfaces, and we ask how much one needs to know about indifference surfaces so as to be able to aggregate individual preferences while respecting the unanimity and anonymity requirements. The introduction of this additional information is formulated in terms of weakening Arrow's axiom of independence of irrelevant alternatives.

The model adopted here is an economic model, namely, the canonical model of division of infinitely divisible commodities among a finite set of agents. We choose to study an economic model rather than the abstract model that is now commonly used in the theory of social choice[1] for two reasons. First, it allows a more fine-grained analysis of information about preferences, because it makes it sensible to talk about marginal rates of substitution and other local notions about indifference surfaces. Second, in an economic model preferences are naturally restricted, and by considering a restricted domain we can hope to obtain positive results with less information than under unrestricted domain.

Our first extension of informational basis is to take account of marginal rates of substitution. It turns out that such infinitesimally local information would not be enough to escape from dictatorship, and we establish an extension of Arrow's theorem. Then, it is natural to take account of the portions of indifference surfaces in some finitely sized neighborhoods of the allocations. Based on this additional information, we can construct a non-dictatorial aggregation rule or *social ordering function*, SOF for short, but still anonymity cannot be attained.

The second direction of extending informational basis focuses on indifference surfaces within the corresponding "Edgeworth box". More precisely, for any two allocations, we define the smallest vector of total resources that makes both allocations feasible, and take the portion of the indifference surface through each allocation in the region below the vector. The introduction of this kind of information, however, does not help us avoid dictatorship.

The third avenue relies on some fixed monotone path from the origin in the consumption space, and focuses on the points of indifference surfaces that belong to this path. The idea of referring to such a monotone path is due to Pazner and Schmeidler [16], and may be justified if the path contains relevant benchmark bundles. Making use of this additional information, and following Pazner and Schmeidler's [16] contribution, we can construct a Paretian and anonymous SOF.

Our final, the largest, extension of informational basis is to take whole indifference surfaces. Given the above result, a Paretian and anonymous SOF can be constructed on this informational basis.

The motivation for our research builds on many works in recent and less recent literature. Attempts to construct SOFs and similar objects embodying unanimity and equity requirements were made by Suzumura [19, 20] and Tadenuma [21]. The idea that information about whole indifference surfaces is sufficient, hinted at by Pazner and Schmeidler [16] and Maniquet [14], was made more precise in Pazner [15] and was revived by

[1]Recollect, however, that Arrow's initial presentations [1; 2] dealt with this economic model of division of commodities.

Bossert, Fleurbaey and Van de gaer [4] and Fleurbaey and Maniquet [8, 9] who were able to construct nicely behaved SOFs on this basis. Campbell and Kelly [5] recently studied essentially the same issue in an abstract model of social choice, and showed that limited information about preferences may be enough. However, their model does not have the rich structure of economic environments, and they focus only on non-dictatorship and do not study how much information is needed for the stronger requirement of anonymity.

The chapter is organized as follows. Section 2 introduces the framework and the main notions. Sections 3-5 consider the four types of extensions of the informational basis of social orderings, and present the results. Section 6 concludes. The appendix contains some proofs.

2 Basic Definitions and Arrow's Theorem

The *population* is fixed. Let $N := \{1, ..., n\}$ be the set of *agents* where $2 \leq n < \infty$. There are ℓ *goods* indexed by $k = 1, ..., \ell$ where $2 \leq \ell < \infty$. Agent i's *consumption bundle* is a vector $x_i := (x_{i1}, ..., x_{i\ell})$. An *allocation* is denoted $x := (x_1, ..., x_n)$. The *set of allocations* is $\mathbb{R}_+^{n\ell}$. The set of allocations such that no individual bundle x_i is equal to the zero vector is denoted X. Vector inequalities are denoted as usual: $\geq, >$, and $\gg$.

A *preordering* is a reflexive and transitive binary relation. Agent i's *preferences* are described by a complete preordering R_i (strict preference P_i, indifference I_i) on $\mathbb{R}_+^{\ell}$. A *profile of preferences* is denoted $\mathbf{R} := (R_1, ..., R_n)$. Let $\mathcal{R}$ be the set of continuous, convex, and strictly monotonic preferences over $\mathbb{R}_+^{\ell}$.

A SOF is a mapping $\bar{R}$ defined on $\mathcal{R}^n$ such that, for all $\mathbf{R} \in \mathcal{R}^n$, $\bar{R}(\mathbf{R})$ is a complete preordering on the set of allocations $\mathbb{R}_+^{n\ell}$. Let $\bar{P}(\mathbf{R})$ (resp. $\bar{I}(\mathbf{R})$) denote the strict preference (resp. indifference) relation associated to $\bar{R}(\mathbf{R})$.

Let π be a bijection on N. For each $x \in \mathbb{R}_+^{n\ell}$, define $\pi(x) := (x_{\pi(1)}, \ldots, x_{\pi(n)}) \in \mathbb{R}_+^{n\ell}$, and for each $\mathbf{R} \in \mathcal{R}^n$, define $\pi(\mathbf{R}) := (R_{\pi(1)}, \ldots, R_{\pi(n)}) \in \mathcal{R}^n$. Let Π be the set of all bijections on N. The basic requirements of unanimity and anonymity on which we focus in this chapter are the following.

Weak Pareto: $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, if $\forall i \in N, x_i P_i y_i$, then $x \bar{P}(\mathbf{R}) y$.

Anonymity: $\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in \mathbb{R}_+^{n\ell}, \forall \pi \in \Pi$:

$$x \bar{R}(\mathbf{R}) y \Leftrightarrow \pi(x) \ \bar{R}(\pi(\mathbf{R})) \ \pi(y).$$

Concerning the non-dictatorship form of anonymity, we only define here what dictatorship means, for convenience. Notice that it has to do only with allocations in X, that is, without the zero bundle for any agent.

Dictatorial SOF: A SOF $\bar{R}$ is dictatorial if there exists $i_0 \in N$ such that:

$$\forall \mathbf{R} \in \mathcal{R}^n, \forall x, y \in X : x_{i_0} P_{i_0} y_{i_0} \Rightarrow x \bar{P}(\mathbf{R}) y.$$

The traditional, Arrovian, version of Independence of Irrelevant Alternatives is:

Independence of Irrelevant Alternatives (IIA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, if $\forall i \in N$, R_i and R_i' agree on $\{x_i, y_i\}$, then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

IIA requires that the social ranking of any pair of allocations depends only on agents' binary preferences *over that pair*. Hence, the informational basis of construction of social orderings is very restricted.

The version of Arrow's theorem for the present canonical model of division of commodities is due to Bordes and Le Breton [3].

Proposition 1 (Bordes and Le Breton [3]) *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA, then it is dictatorial.*

3 Local Extension of Informational Basis

The IIA axiom can be weakened by strengthening the premise: that is, for any two preference profiles and any pair of allocations, only when some properties about indifference surfaces associated with the two allocations coincide in addition to pairwise preferences, it is required that the social ranking over the two allocations should agree. This amounts to allowing the SOF to make use of more information about indifference surfaces when ranking each pair of allocations.

In this chapter, we consider four types of extensions of the informational basis of social orderings. First, we use information about marginal rates of substitution. Economists are used to focus on marginal rates of substitution when assessing the efficiency of an allocation, especially under convexity, since for convex preferences the marginal rates of substitution determine the half space in which the upper contour set lies. Moreover, for efficient allocations, the shadow prices enable one to compute the relative implicit income shares of different agents, thereby potentially providing a relevant measure of inequalities in the distribution of resources. Therefore, taking account of marginal rates of substitution is a natural extension of the informational basis of social choice in economic environments.

Let $C(x_i, R_i)$ denote the cone of price vectors that support the upper contour set for R_i at x_i :

$$C(x_i, R_i) := \{p \in \mathbb{R}^\ell | \forall y \in \mathbb{R}_+^\ell, py = px_i \Rightarrow x_i R_i y\}.$$

When preferences R_i are strictly monotonic, one has $C(x_i, R_i) \subset \mathbb{R}_{++}^\ell$ whenever $x_i \gg 0$.

IIA except Marginal Rates of Substitution (IIA-MRS): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, if $\forall i \in N$, R_i and R_i' agree on $\{x_i, y_i\}$, and

$$\begin{aligned} C(x_i, R_i) &= C(x_i, R_i'), \\ C(y_i, R_i) &= C(y_i, R_i'), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

It is clear that IIA implies IIA-MRS. The converse does not hold as an example in Appendix A.4 shows. It turns out, unfortunately, that weakening IIA into IIA-MRS cannot alter the dictatorship conclusion of Arrow's theorem. Introducing information about marginal rates of substitution, in addition to pairwise preferences, does not make room for satisfactory SOFs.

Proposition 2 *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA-MRS, then it is dictatorial.*

The proof of Proposition 2 is long and is relegated to the appendix, but here we sketch the main line of the proof. Since IIA implies IIA-MRS, Proposition 2 is a generalization of the theorem by Bordes and Le Breton [3, Theorem 3]. An essential idea of the proofs of Arrow-like theorems in economic environments (Kalai, Muller and Satterthwaite [13], Bordes and Le Breton [3], and others) is as follows: First, we find a "free triple", that is, three allocations for which any ranking is possible in each individual's preferences satisfying the standard assumptions in economics. By applying Arrow's theorem for these three allocations, it can be shown that there exists a "local dictator" for each free triple. Then, we "connect" free triples in a suitable way to show that these local dictators must be the same individual.

Turning to IIA-MRS, notice first that for each free triple, IIA-MRS works just as IIA only in the class of preference profiles for which individuals' marginal rates of substitution at the three allocations do not change from one profile to another, and satisfy certain "supporting conditions". Invoking Arrow's theorem, we can only show that there exists a "local dictator" for each free triple in this much restricted class of preference profiles (Lemmas A.1 and A.2). The difficulty in the proof of Proposition 2 lies in extending "local dictatorship" over the class of all preference profiles. This requires much work to do. See Lemmas A.3 and A.4 in the Appendix.

Inada [12] also considered marginal rates of substitution in an IIA-like axiom, but the difference from our work is that he looked for a local aggregator of preferences, namely a mapping defining a social marginal rate of substitution between goods and individuals, on the basis of individual marginal rates of substitution. Hence, Inada requires that, for each allocation, social preferences *in an infinitely small neighborhood of the allocation* should not change whenever every agent's marginal rates of substitution at the allocation remain the same. By contrast, our IIA-MRS requires that, for each pair of allocations, social preferences *over that pair* should not change whenever every agent's marginal rates of substitution at each of the two allocations remain the same. There is no logical relation between Inada's axiom and ours.

Marginal rates of substitution give an infinitesimally local piece of information about indifference surfaces at given allocations. A natural extension of the informational basis would be to take account of the indifference surfaces in some finitely sized neighborhoods of the two allocations. Define, for any given real number $\varepsilon > 0$,

$$B_\varepsilon(x_i) := \{v \in \mathbb{R}^\ell_+ | \max_{k \in \{1,\dots,\ell\}} |x_{ik} - v_k| \leq \varepsilon\}.$$

Define

$$I(x_i, R_i) := \{z \in \mathbb{R}^\ell_+ \mid z \; I_i \; x_i\}.$$

The set $I(x_i, R_i)$ is called the *indifference surface* at x_i for R_i.

The next axiom is defined for any given $\varepsilon > 0$.

IIA except Indifference Surfaces in ε-Neighborhoods (IIA-ISεN): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$, R_i and R'_i agree on $\{x_i, y_i\}$, and

$$\begin{aligned} I(x_i, R_i) \cap B_\varepsilon(x_i) &= I(x_i, R'_i) \cap B_\varepsilon(x_i), \\ I(y_i, R_i) \cap B_\varepsilon(y_i) &= I(y_i, R'_i) \cap B_\varepsilon(y_i), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

It is clear that for any given $\varepsilon > 0$, IIA-MRS implies IIA-ISεN. Notice also that the larger is the value of ε, the weaker the condition IIA-ISεN becomes.

The next proposition shows that as soon as one switches from IIA-MRS to IIA-ISεN, the dictatorship result is avoided *even if ε is arbitrarily small.* However, it remains impossible to achieve Anonymity *even for an arbitrarily large ε.*

Proposition 3 *For any given $\varepsilon > 0$, there exists a SOF that satisfies Weak Pareto, IIA-ISεN, and is not dictatorial. However, for any given $\varepsilon > 0$, there exists no SOF that satisfies Weak Pareto, IIA-ISεN and Anonymity.*

Proof. The proof of the impossibility part is in the appendix. Here we prove the possibility part. Define $\bar{R}$ as follows: $x\bar{R}(\mathbf{R})y$ if either $x_1 R_1 y_1$ and $[I(x_1, R_1) \nsubseteq B_\varepsilon(0)$ or $I(y_1, R_1) \nsubseteq B_\varepsilon(0)]$, or $x_2 R_2 y_2$ and $[I(x_1, R_1) \subseteq B_\varepsilon(0)$ and $I(y_1, R_1) \subseteq B_\varepsilon(0)]$. For brevity, let $\Gamma(v)$ denote $[I(v, R_1) \subseteq B_\varepsilon(0)]$. Weak Pareto and the absence of dictator are straightforwardly satisfied. IIA-ISεN is also satisfied because when $\Gamma(x_1)$ and $\Gamma(y_1)$ hold, we have $B_\varepsilon(0) \subseteq B_\varepsilon(x_1) \cap B_\varepsilon(y_1)$, and therefore $\Gamma(x_1)$ and $\Gamma(y_1)$ remain true if the indifference surfaces are kept fixed on $B_\varepsilon(x_1)$ and $B_\varepsilon(y_1)$. It remains to check transitivity of $\bar{R}(\mathbf{R})$. First, note the following property: If $\Gamma(v)$ holds and vR_1v', then $\Gamma(v')$ also holds. Assume that there exist $x, y, z \in \mathbb{R}^{n\ell}_+$ such that $x\bar{R}(\mathbf{R})y\bar{R}(\mathbf{R})z\bar{P}(\mathbf{R})x$. If $\Gamma(x_1)$, $\Gamma(y_1)$ and $\Gamma(z_1)$ all hold, this is impossible because one should have $x_2R_2y_2R_2z_2P_2x_2$. If only one of the three conditions $\Gamma(x_1)$, $\Gamma(y_1)$, $\Gamma(z_1)$ is satisfied, it is similarly impossible because one should have $x_1R_1y_1R_1z_1P_1x_1$. Assume $\Gamma(x_1)$ and $\Gamma(y_1)$ hold, but not $\Gamma(z_1)$. Then, $y\bar{R}(\mathbf{R})z\bar{P}(\mathbf{R})x$ requires $y_1R_1z_1P_1x_1$, which implies $\Gamma(z_1)$, a contradiction. Assume $\Gamma(x_1)$ and $\Gamma(z_1)$ hold, but not $\Gamma(y_1)$. Then, $x\bar{R}(\mathbf{R})y\bar{R}(\mathbf{R})z$ requires $x_1R_1y_1R_1z_1$, which implies $\Gamma(y_1)$, a contradiction. Assume $\Gamma(y_1)$ and $\Gamma(z_1)$ hold, but not $\Gamma(x_1)$. Then, $z\bar{P}(\mathbf{R})x\bar{R}(\mathbf{R})y$ requires $z_1P_1x_1R_1y_1$, which implies $\Gamma(x_1)$, a contradiction. ■

4 Extension of Informational Basis to "Edgeworth Boxes"

Our second type of extension of informational basis is to focus on the portions of indifference surfaces, associated with each pair of allocations, that lie within the corresponding

"Edgeworth box": namely, the set of bundles that are achievable by redistributing the two allocations under consideration. However, for a given pair of allocations, the two allocations may need different amounts of total resources to be feasible. Therefore we need to introduce the following notions. For each good $k \in \{1, ..., \ell\}$, define

$$\omega_k(x, y) := \max \{\sum_{i \in N} x_{ik}, \sum_{i \in N} y_{ik}\}.$$

Let $\omega(x, y) := (\omega_1(x, y), ..., \omega_\ell(x, y))$. The vector $\omega(x, y) \in \mathbb{R}^\ell_+$ represents the smallest amount of total resources that makes two allocations x and y feasible. Fig. 1 illustrates the construction of $\omega(x, y)$. Then, define

$$\Omega(x, y) := \left\{ z \in \mathbb{R}^\ell_+ \mid z \leq \omega(x, y) \right\}.$$

The set $\Omega(x, y) \subset \mathbb{R}^\ell_+$ is the set of consumption bundles that are feasible with $\omega(x, y)$. The following axiom captures the idea that the ranking of any two allocations should depend only on the indifference surfaces over the region satisfying the corresponding feasibility constraint.

IIA except Indifference Surfaces over Feasible Allocations (IIA-ISFA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Omega(x, y) &= I(x_i, R'_i) \cap \Omega(x, y), \\ I(y_i, R_i) \cap \Omega(x, y) &= I(y_i, R'_i) \cap \Omega(x, y), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

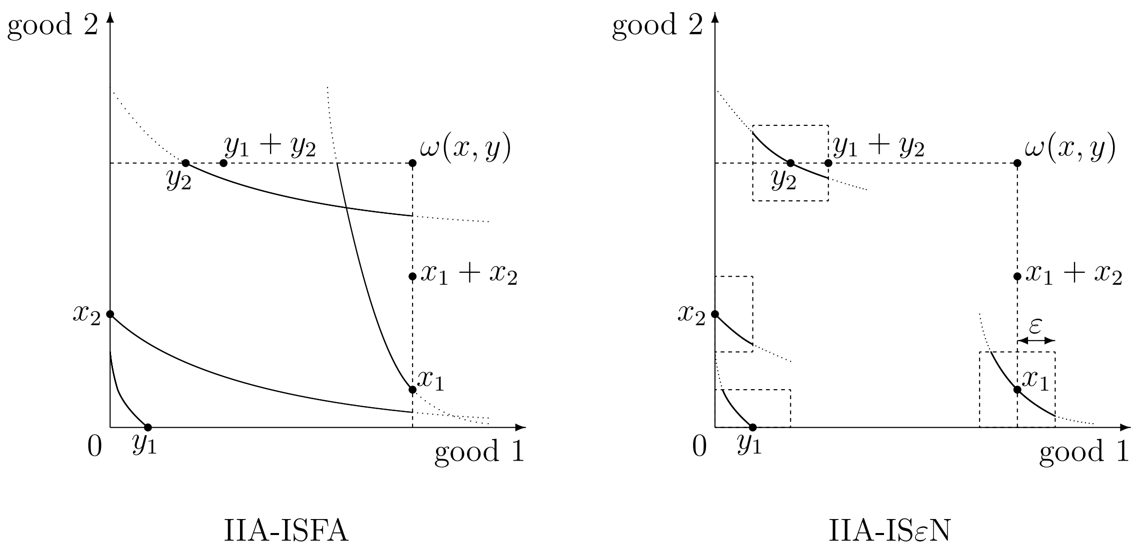

Figure 1: Relevant Portions of Indifference Surfaces under IIA-ISFA and IIA-ISεN

The left figure in Figure 1 illustrates the relevant portions of indifference surfaces under IIA-ISFA, which are indicated by thick curves, while the right figure shows the relevant parts under IIA-ISεN. One can see that, in general, there is no inclusion relation between the relevant portions of indifference surfaces under IIA-ISFA and those under IIA-ISεN. Hence, there is no logical relation between the axioms IIA-ISFA and IIA-ISεN. See counterexamples in Appendix A.4.

The introduction of information about indifference surfaces over the region satisfying the corresponding feasibility constraint, however, cannot help us avoid a dictatorial SOF.

Proposition 4 *If a SOF satisfies Weak Pareto and IIA-ISFA, then it is dictatorial.*

The proof relies on the following lemmas. First, we define a weak form of IIA:

Weak Independence of Irrelevant Alternatives (WIIA): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in X$, if $\forall i \in N$, R_i and R'_i agree on $\{x_i, y_i\}$, and for no i, $x_i \; I_i \; y_i$, then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

A key lemma to prove Proposition 4 is the following:

Lemma 1 *If a SOF $\bar{R}$ satisfies Weak Pareto and IIA-ISFA, then it satisfies WIIA.*

The proof of this lemma is long and relegated in the appendix. We also define a weak form of dictatorship: Given a SOF $\bar{R}$, $Y \subseteq X$ and $\mathcal{R}' \subseteq \mathcal{R}^n$, we say that agent $i_0 \in N$ is a *quasi-dictator for $\bar{R}$ over* $(Y, \mathcal{R}')$ if for all $x, y \in Y$, and all $\mathbf{R} \in \mathcal{R}'$, whenever $x_{i_0} P_{i_0} y_{i_0}$ and there is no $i \in N$ with $x_i I_i y_i$, we have $x \bar{P}(\mathbf{R}) y$.

Quasi-Dictatorial SOF: A SOF $\bar{R}$ is quasi-dictatorial if there exists a quasi-dictator $i_0 \in N$ for $\bar{R}$ over $(X, \mathcal{R}^n)$.

Lemma 2 *If a SOF $\bar{R}$ satisfies Weak Pareto and WIIA, then it is quasi-dictatorial.*

Proof. Let $\bar{R}$ be a SOF that satisfies Weak Pareto and WIIA. By an adaptation of a standard proof of Arrow's theorem (for instance, Sen [18]), we can show that for every free triple $Y \subset X$, there exists a quasi-dictator over $(Y, \mathcal{R}^n)$. Then, a direct application of Bordes and Le Breton [3] establishes quasi-dictatorship of $\bar{R}$. ■

It is interesting that, in our economic environments, quasi-dictatorship is equivalent to dictatorship as the next lemma shows.

Lemma 3 *If a SOF $\bar{R}$ is quasi-dictatorial, then it is dictatorial.*

Proof. Let $\bar{R}$ be a quasi-dictatorial SOF. Let $x, y \in X$ and $\mathbf{R} \in \mathcal{R}^n$ be such that $x_{i_0} P_{i_0} y_{i_0}$. By continuity and strict monotonicity of preferences, there exists $x' \in X$ such that $x_{i_0} P_{i_0} x'_{i_0} P_{i_0} y_{i_0}$ and for all $i \in N$, either $x_i P_i x'_i P_i y_i$ or $y_i R_i x_i P_i x'_i$. Since $\bar{R}$ is quasi-dictatorial, it follows that $x \bar{P}(\mathbf{R}) x'$ and $x' \bar{P}(\mathbf{R}) y$. By transitivity, $x \bar{P}(\mathbf{R}) y$. ■

Given these lemmas, the proof of Proposition 4 is straightforward.

Proof of Proposition 4: Let $\bar{R}$ be a SOF that satisfies Weak Pareto and IIA-ISFA. By Lemma 1, $\bar{R}$ satisfies WIIA. Then, by Lemmas 2 and 3, $\bar{R}$ is dictatorial. ■

One may define a weaker axiom than IIA-ISFA by considering a radial expansion of the corresponding "Edgeworth box", namely $\lambda\Omega(x, y)$ for a given $\lambda \geq 1$, where λ can be arbitrarily large. With this version, however, the same impossibility still holds as Proposition 4.

5 Extension of Informational Basis with a Monotone Path

The previous sections have shown that non-local information about indifference surfaces is needed to construct a satisfactory SOF. This does not mean, however, that a lot of information is needed. In this section we show that knowing one point in each indifference surface may be enough.

Our third way of extending information about indifference surfaces is to rely on a path

$$\Lambda_{\omega_0} := \{\lambda\omega_0 \in \mathbb{R}^{\ell}_{++} \mid \lambda \in \mathbb{R}_+\},$$

where $\omega_0 \in \mathbb{R}^{\ell}_{++}$ is fixed, and to focus on the point of each indifference surface that belongs to this path. The idea of referring to such a path is due to Pazner and Schmeidler [16], and may be justified if the path contains relevant benchmark bundles. Although the choice of ω_0 is not discussed here, it need not be arbitrary. For instance, one may imagine that it could reflect an appropriate equity notion, or it could be the bundle of the total available resources.

IIA except Indifference Surfaces on Path $\boldsymbol{\omega_0}$ (IIA-ISP$\boldsymbol{\omega_0}$): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_+$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Lambda_{\omega_0} &= I(x_i, R'_i) \cap \Lambda_{\omega_0}, \\ I(y_i, R_i) \cap \Lambda_{\omega_0} &= I(y_i, R'_i) \cap \Lambda_{\omega_0}, \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

Following Pazner and Schmeidler's [16] contribution, we can derive the next result, which means that not much information is needed to have an anonymous SOF if only we are prepared to accept an externally specified reference bundle.

Proposition 5 *For any given $\omega_0 \in \mathbb{R}^{\ell}_{++}$, there exists a SOF that satisfies Weak Pareto, IIA-ISPω_0 and Anonymity.*

Proof. For each $i \in N$, each $R_i \in \mathcal{R}$ and each $x_i \in \mathbb{R}^{\ell}_{+}$, let $\alpha(x_i, R_i) \in \mathbb{R}_+$ be the scalar such that $\alpha(x_i, R_i)\omega_0 I_i x_i$. By continuity and strict monotonicity of preferences, $\alpha(x_i, R_i)$ always exists uniquely. Let $\bar{R}$ be defined by:

$$x\bar{R}(\mathbf{R})y \Leftrightarrow \min_{i \in N} \alpha(x_i, R_i) \geq \min_{i \in N} \alpha(y_i, R_i).$$

This SOF clearly satisfies Weak Pareto and Anonymity. It also satisfies IIA-ISPω_0 because whenever $I(x_i, R_i) \cap \Lambda_{\omega_0} = I(x_i, R'_i) \cap \Lambda_{\omega_0}$, we have $\alpha(x_i, R_i) = \alpha(x_i, R'_i)$. ■

By relying on the leximin criterion rather than the maximin for the SOF defined in the above proof, we could have the **Strong Pareto** property as well: $\forall x, y \in \mathbb{R}^{n\ell}_{+}, \forall \mathbf{R} \in \mathcal{R}^n$ if $\forall i \in N,\ x_i R_i y_i$, then $x\bar{R}(\mathbf{R})y$ and if, in addition, $\exists i \in N,\ x_i P_i y_i$, then $x\bar{P}(\mathbf{R})y$.

The final extension of informational basis that we consider is to introduce whole indifference surfaces. This condition was already introduced and studied by Hansson [11] in the abstract model of social choice, who showed that the Borda rule, which does not satisfy the Arrow IIA condition, satisfies this constrained variant thereof. Pazner [15] also proposed it, in a study of social choice in economic environments.

IIA except Whole Indifference Surfaces (IIA-WIS): $\forall \mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}^{n\ell}_{+}$, if $\forall i \in N$,

$$\begin{aligned} I(x_i, R_i) &= I(x_i, R'_i), \\ I(y_i, R_i) &= I(y_i, R'_i), \end{aligned}$$

then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$.

Since IIA-ISPω_0 (as well as every other IIA type axiom introduced so far) implies IIA-WIS, we have the following corollary.

Corollary 1 *There exists a SOF that satisfies Weak Pareto, IIA-WIS and Anonymity.*

There are many examples of SOFs satisfying Weak Pareto, IIA-WIS and Anonymity. Thus, in addition to these three axioms, we may add other requirements embodying various equity principles. Notice that Strong Pareto and Anonymity already entail a version of *the Suppes grading principle*: for all $\mathbf{R} \in \mathcal{R}^n$, all x, y, if there are i, j such that $R_i = R_j$, $x_i P_i y_j$ and $x_j P_i y_i$, and for $h \neq i, j$, $x_h = y_h$, then $x\bar{P}(\mathbf{R})y$. We can also construct SOFs satisfying Strong Pareto, IIA-WIS (or IIA-ISPω_0), Anonymity and the following version of *the Hammond equity axiom* (Hammond [10]): for all $\mathbf{R} \in \mathcal{R}^n$, and all $x, y \in \mathbb{R}^{n\ell}_{+}$, if there are i, j such that $R_i = R_j$, $y_i P_i x_i P_i x_j P_i y_j$, and for all $h \neq i, j$, $x_h = y_h$, then $x\bar{P}(\mathbf{R})y$.

Let us summarize in Figure 2 the various IIA type axioms that we have introduced, and the main results in this chapter. The arrows indicate logical relations between the axioms. Weaker axioms allow SOFs to depend on more information about indifference surfaces. In the appendix we show that all the implications are strict (the converse relations do not hold). The dotted lines in the figure indicate borderlines between possibility and impossibility, under Weak Pareto, of Non-Dictatorship and of Anonymity.

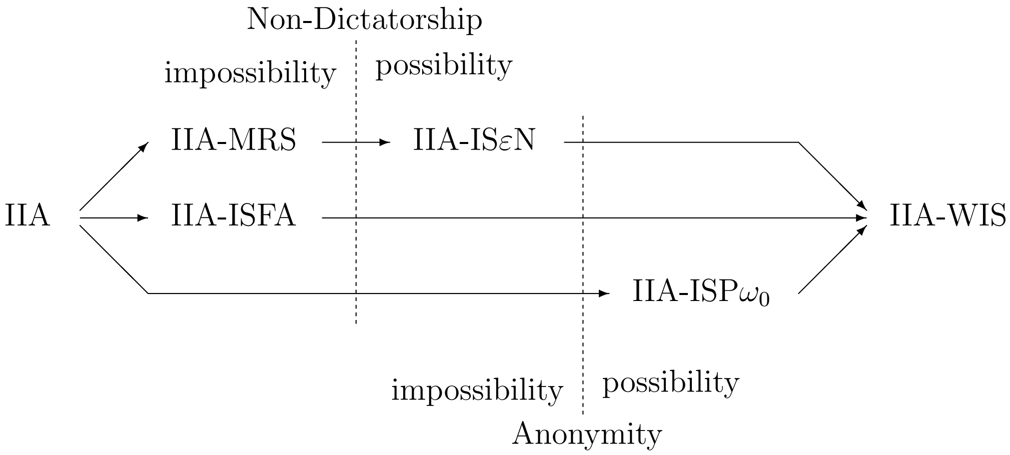

Figure 2: Various IIA Axioms and Summary of the Main Results

6 Conclusion

The construction of a non-dictatorial Arrovian SOF in a framework with purely ordinal, interpersonally non-comparable preferences requires information about the shape of indifference surfaces that goes well beyond infinitesimally local data such as marginal rates of substitution or data within the corresponding "Edgeworth box". On the basis of information in some finitely sized neighborhoods, one can construct a non-dictatorial SOF, but still cannot have an anonymous one. Only substantially non-local information about indifference surfaces enables one to construct a Paretian and anonymous SOF. These are the main messages of this chapter, in which we proved two extensions of Arrow's impossibility theorem, and several possibility results. We hope that this chapter, more broadly, contributes to clarifying the informational foundations in the theory of social choice.

There are limits to our work which may be noticed here, and call for further research. First, we study a particular economic model, and it would be worth analyzing the same issues in other models such as the standard abstract model of social choice or other economic models, in particular models with public goods (the case of consumption externalities in our model could also be subsumed under the case of public goods). Second, the information about indifference surfaces is a complex set of objects, and our analysis is far from being exhaustive on the pieces of data which can be extracted from this set. We have focussed on what seemed to us the most natural parts of indifference surfaces to which one may want to refer in social evaluation of allocations, namely, the marginal rates of substitution, the Edgeworth box (bundles which are achievable by redistributing the considered allocations), and reference rays. But there may be other ways of considering indifference surfaces. For instance, it would be nice to have a measure of the degree to which a given piece of information is local, and the connection between this work and topological social choice (e.g. Chichilnisky [6]) might be worth exploring. Third, there may be other kinds of interesting additional information. For instance, Roberts [17] considered introducing information about utilities and about non-local preferences at the same time, and was able to characterize the Nash social welfare function on this basis. There certainly are many avenues of research along these lines. The purpose of this chapter would be well-served if it could open the gate towards these enticing avenues.

Appendix

A.1 Proof of Proposition 2

The proof of Proposition 2 relies on the following lemmas.

Let $Y \subset X$ be a given finite subset of X. Let $i \in N$ be given. Define $Y_i := \{y_i \in \mathbb{R}^\ell_+ | \exists y_{-i} \in \mathbb{R}^{(n-1)\ell}_+, (y_i, y_{-i}) \in Y\}$. Let $\mathcal{Q}$ denote the set of convex cones in $\mathbb{R}^\ell_{++}$. For each $y_i \in Y_i$, let $Q(y_i) \in \mathcal{Q}$ be given. We say that the set Y_i *satisfies the supporting condition with respect to* $\{Q(y_i)|y_i \in Y_i\}$ if for all $y_i \in Y_i$, all $q \in Q(y_i)$, and all $y'_i \in Y_i$ with $y'_i \neq y_i$, $q \cdot y_i < q \cdot y'_i$. Define

$$\mathcal{R}(Y_i, \{Q(y_i) \mid y_i \in Y_i\}) := \{R_i \in \mathcal{R} \mid \forall y_i \in Y_i, C(y_i, R_i) = Q(y_i)\}.$$

The set of all complete preorderings on Y_i is denoted by $\mathcal{O}(Y_i)$. For all $R_i \in \mathcal{R}$, $R_i|_{Y_i}$ denotes the restriction of R_i on Y_i. Namely, $R_i|_{Y_i}$ is the complete preordering on Y_i such that for all $x_i, y_i \in Y_i$, $x_i \; R_i|_{Y_i} \; y_i \Leftrightarrow x_i \; R_i \; y_i$. For all $\mathcal{R}' \subset \mathcal{R}$, let $\mathcal{R}'|_{Y_i} := \{R_i|_{Y_i} \mid R_i \in \mathcal{R}'\}$. For all $x_i \in X$ and all $R_i \in \mathcal{R}$, let $U(x_i, R_i) := \{x'_i \in X \mid x'_i \; R_i \; x_i\}$ denote the (closed) upper contour set of x_i for R_i.

Lemma A.1 *If a finite set $Y_i \subset \mathbb{R}^\ell_+$ satisfies the supporting condition with respect to $\{Q(y_i)|y_i \in Y_i\}$, then $\mathcal{R}(Y_i, \{Q(y_i)|y_i \in Y_i\})|_{Y_i} = \mathcal{O}(Y_i)$.*

Proof. We have only to show that $\mathcal{O}(Y_i) \subseteq \mathcal{R}(Y_i, \{Q(y_i)|y_i \in Y_i\})|_{Y_i}$. Let $R' \in \mathcal{O}(Y_i)$ be any preordering on Y_i. Construct a complete preordering $R_i \in \mathcal{R}$ so that the upper contour set of each $y_i \in Y_i$ is defined as follows. Let $x_i \in Y_i$ be such that for all $y_i \in Y_i$, $y_i R'_i x_i$. Define $Y_i^1 := \{y_i \in Y_i | y_i I'_i x_i\}$. For each $a \in \mathbb{R}^\ell_+$ and each $q \in \mathbb{R}^\ell_{++}$, define $H(a, q) := \{b \in \mathbb{R}^\ell_+ | q \cdot b \geq q \cdot a\}$. Let

$$U(x_i, R_i) := \bigcap_{y_i \in Y_i^1} \left[\bigcap_{q \in Q(y_i)} H(y_i, q) \right]$$

Let $I(x_i, R_i)$ be the boundary of $U(x_i, R_i)$. Clearly, for all $y_i \in Y_i^1$, $C(y_i, R_i) = Q(y_i)$. We also have that, for all $y_i \in Y_i \backslash Y_i^1$, and for all $x'_i \in I(x_i, R_i)$, $y_i P_i x'_i$. Given $\delta > 0$, let $(1+\delta)U(x_i, R_i) := \{x'_i \in \mathbb{R}^\ell_+ | \exists a_i \in U(x_i, R_i), x'_i = (1+\delta)a_i\}$, and let $(1+\delta)I(x_i, R_i)$ be the boundary of $(1+\delta)U(x_i, R_i)$. For sufficiently small δ, we have that for all $y_i \in Y_i \backslash Y_i^1$

and all $x_i' \in (1+\delta)I(x_i, R_i)$, $y_i P_i x_i'$ by continuity of preferences. Let $z_i \in Y_i \backslash Y_i^1$ be such that for all $y_i \in Y_i \backslash Y_i^1$, $y_i R_i' z_i$. Define $Y_i^2 := \{y_i \in Y_i \backslash Y_i^1 | y_i I_i' z_i\}$. Let

$$U(z_i, R_i) := (1+\delta)U(x_i, R_i) \bigcap \left(\bigcap_{y_i \in Y_i^2} \left[\bigcap_{q \in Q(y_i)} H(y_i, q) \right] \right)$$

Let $I(z_i, R_i)$ be the boundary of $U(z_i, R_i)$. By definition, for all $y_i \in Y_i^2$, $C(y_i, R_i) = Q(y_i)$. We have that, for all $y_i \in Y_i \backslash (Y_i^1 \cup Y_i^2)$ and all $x_i' \in I(z_i, R_i)$, $y_i P_i x_i'$. Similarly we can construct the upper contour set of each $y_i \in Y_i \backslash (Y_i^1 \cup Y_i^2)$. By its construction, $R_i \in \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})$ and $R_i|_{Y_i} = R'$. Thus, $R' \in \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})|_{Y_i}$. ■

Let $\bar{R}$ be a SOF. Let $Y \subseteq X$ and $\mathcal{R}' \subseteq \mathcal{R}^n$ be given. We say that agent $i_0 \in N$ is a *local dictator for $\bar{R}$ over* $(Y, \mathcal{R}')$ if, for all $x, y \in Y$ and all $\mathbf{R} \in \mathcal{R}'$, $x_{i_0} P_{i_0} y_{i_0}$ implies $x \bar{P}(\mathbf{R}) y$.

Given a set A, let $|A|$ denote the cardinality of A.

Lemma A.2 *Let $\bar{R}$ be a SOF satisfying Weak Pareto and IIA-MRS. Let $Y \subset X$ be a finite subset of X such that $|Y| \geq 3$. Suppose that for all $i \in N$, Y_i satisfies the supporting condition with respect to $\{Q(y_i) | y_i \in Y_i\}$. Then, there exists a local dictator $i_0 \in N$ for $\bar{R}$ over $(Y, \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\}))$.*

Proof. For all $\mathbf{R}, \mathbf{R}' \in \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})$, all $y \in Y$, and all $i \in N$, $C(y_i, R_i) = C(y_i, R_i')$. Since $\bar{R}$ satisfies IIA-MRS, we have that, for all $x, y \in Y$, and all $\mathbf{R}, \mathbf{R}' \in \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) | \, y_i \in Y_i\})$, if R_i and R_i' agree on $\{x_i, y_i\}$ for all $i \in N$, then $\bar{R}(\mathbf{R})$ and $\bar{R}(\mathbf{R}')$ agree on $\{x, y\}$. By Lemma A.1, for all $i \in N, \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\})|_{Y_i} = \mathcal{O}(Y_i)$. Hence, by Arrow's Theorem, there exists a local dictator for $\bar{R}$ over $(Y, \prod_{i \in N} \mathcal{R}(Y_i, \{Q(y_i) | y_i \in Y_i\}))$. ■

We say that a subset Y of X is *free for agent i* if $\mathcal{R}|_{Y_i} = \mathcal{O}(Y_i)$. It is *free* if it is free for all $i \in N$. If Y contains two elements, it is a *free pair*. If Y contains three elements, it is a *free triple*. Note that a set $\{x, y\}$ is a free pair for $i \in N$ if and only if, for some $k, k' \in \{1, \ldots, \ell\}$, $x_{ik} > y_{ik}$ and $y_{ik'} > x_{ik'}$. Given two consumption bundles $x_i, y_i \in \mathbb{R}_+^\ell$, define $x_i \wedge y_i \in \mathbb{R}_+^\ell$ as $(x_i \wedge y_i)_k = \min\{x_{ik}, y_{ik}\}$ for all $k \in \{1, \ldots, \ell\}$.

Lemma A.3 *Let $\bar{R}$ be a SOF satisfying Weak Pareto and IIA-MRS. If $\{x, y\} \subset X$ is a free pair, then there exists a local dictator for $\bar{R}$ over $(\{x, y\}, \mathcal{R}^n)$.*

Proof. Let $\bar{R}$ be a SOF satisfying Weak Pareto and IIA-MRS. Let $\{x, y\} \subset X$ be a free pair. Let

$$\begin{aligned} K_1 &:= \{k \in \{1, \ldots, \ell\} \mid x_{ik} > y_{ik}\} \\ K_2 &:= \{k \in \{1, \ldots, \ell\} \mid x_{ik} < y_{ik}\} \end{aligned}$$

Since $\{x, y\}$ is a free pair, $K_1, K_2 \neq \emptyset$.

Step 1 : For each $i \in N$, we define two consumption bundles $z_i, w_i \in X$ as follows:

$$z_i := x_i \wedge y_i + \frac{1}{2}\left[\frac{2}{3}(x_i - x_i \wedge y_i) + \frac{1}{3}(y_i - x_i \wedge y_i)\right] \tag{1}$$

$$w_i := x_i \wedge y_i + \frac{1}{2}\left[\frac{1}{3}(x_i - x_i \wedge y_i) + \frac{2}{3}(y_i - x_i \wedge y_i)\right] \tag{2}$$

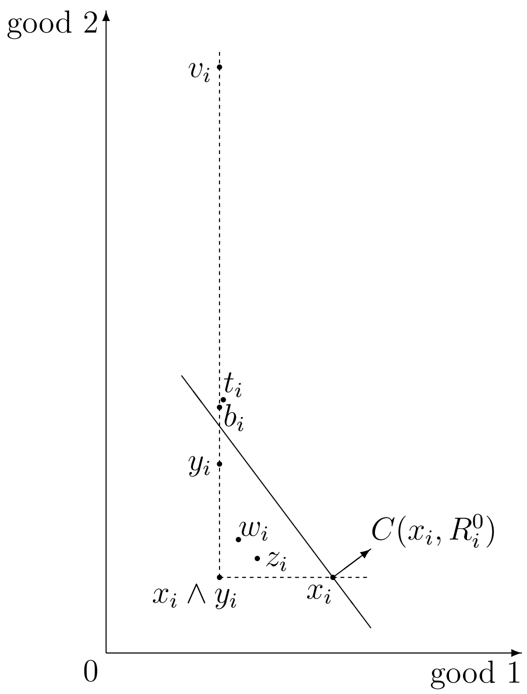

Figure 3: Proof of Lemma A.3

Figure 3 illustrates the bundles $x_i, y_i, x_i \wedge y_i, z_i, w_i$, and also b_i, v_i, t_i, which are defined in the next step. Let $q \in \mathbb{R}^{\ell}_{++}$. Then, $q \cdot y_i < q \cdot w_i$ if and only if

$$\frac{2}{3}\sum_{k \in K_2} q_k(y_{ik} - x_{ik}) < \frac{1}{6}\sum_{k \in K_1} q_k(x_{ik} - y_{ik}) \tag{3}$$

Since $K_1 \neq \emptyset$, the right-hand-side of (3) can be arbitrarily large as $(q_k)_{k \in K_1}$ become large, $(q_k)_{k \in K_2}$ being constant. Hence, there exists a price vector $q(y_i) \in \mathbb{R}^{\ell}_{++}$ that satisfies inequality (3). With some calculations, it can be shown that $q(y_i) \cdot y_i < q(y_i) \cdot z_i$ and $q(y_i) \cdot y_i < q(y_i) \cdot x_i$.

Similarly, for each $a \in \{x_i, z_i, w_i\}$, we can find a price vector $q(a) \in \mathbb{R}^{\ell}_{++}$ such that, for all $a' \in \{x_i, z_i, w_i, y_i\}$ with $a' \neq a$, $q(a) \cdot a < q(a) \cdot a'$. Hence, the set $Y_i^0 = \{x_i, z_i, w_i, y_i\}$ satisfies the supporting condition with respect to $\{q(x_i), q(z_i), q(w_i), q(y_i)\}$.[2]

[2]With a slight abuse of notation, we write $q(\cdot)$ for $Q(\cdot) = \{\alpha q(\cdot) \mid \alpha > 0\}$.

Let $z := (z_i)_{i \in N}$ and $w := (w_i)_{i \in N}$. Let $Y^0 := \{x, z, w, y\}$. By Lemma A.2, there exists a local dictator $i_0 \in N$ for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\}))$.
Step 2: We will show that agent i_0 is a local dictator for $\bar{R}$ over $(\{x, y\}, \mathcal{R}^n)$.

Suppose, on the contrary, that there exists a preference profile $\mathbf{R}^0 \in \mathcal{R}^n$ such that (i) $x_{i_0} P_{i_0}^0 y_{i_0}$ and $y \bar{R}(\mathbf{R}^0) x$ or (ii) $y_{i_0} P_{i_0}^0 x_{i_0}$ and $x \bar{R}(\mathbf{R}^0) y$. Without loss of generality, suppose that (i) holds. Let $Y^1 := \{z, w, y\}$. Since agent i_0 is the local dictator for $\bar{R}$ over $(Y^0, \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\}))$, he is also the local dictator for $\bar{R}$ over $(Y^1, \prod_{i \in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\}))$. (Otherwise, by Lemma A.2, there exists a local dictator $j \neq i_0$ for $\bar{R}$ over $(Y^1, \prod_{i \in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\}))$, and we can construct a preference profile $\mathbf{R} \in \prod_{i \in N} \mathcal{R}(Y_i^0, \{q(x_i), q(z_i), q(w_i), q(y_i)\}) \subset \prod_{i \in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$ such that $z_{i_0} P_{i_0} w_{i_0}$ and $w_j P_j z_j$. Hence we must have $z \bar{P}(\mathbf{R}) w$ and $w \bar{P}(\mathbf{R}) z$, which is a contradiction.)

We define two allocations $v, t \in X$ in the following steps. Let $i \in N$. First, define $b_i \in \mathbb{R}_+^\ell$ as follows: If, for all $q \in C(x_i, R_i^0)$, $q \cdot (y_i - x_i) \geq 0$, then let $b_i := y_i$. If, for some $q \in C(x_i, R_i^0)$, $q \cdot (y_i - x_i) < 0$, then let $\theta > 0$ be a positive number such that, for all $q \in C(x_i, R_i^0)$, $q \cdot [y_i + \theta(y_i - x_i \wedge y_i) - x_i] > 0$. Since $q \in \mathbb{R}_{++}^\ell$ by strict monotonicity of preferences, and $y_i - x_i \wedge y_i > 0$, such a number θ exists. Then, define $b_i := y_i + \theta(y_i - x_i \wedge y_i)$. By definition, $b_i > y_i$, and, for all $q \in C(x_i, R_i^0)$, $q \cdot (b_i - x_i) > 0$. Define

$$v_i := b_i + 2(b_i - x_i \wedge y_i).$$

Then, $v_i > b_i > y_i$, and, for all $q \in C(x_i, R_i^0)$, $q \cdot (v_i - x_i) > 0$.

Next, define

$$t_i := x_i \wedge y_i + \frac{1}{2} \left[\frac{2}{3}(v_i - x_i \wedge y_i) + \frac{1}{3}(w_i - x_i \wedge y_i) \right].$$

Then,

$$t_i = b_i + \frac{1}{6}(w_i - x_i \wedge y_i) > b_i$$

and, for all $q \in C(x_i, R_i^0)$, $q \cdot x_i < q \cdot t_i$.

As in Step 1, we can find price vectors $q(v_i), q(t_i) \in \mathbb{R}_{++}^\ell$ such that $q(v_i) \cdot v_i < q(v_i) \cdot a$ for all $a \in \{x_i, z_i, w_i, t_i\}$, and $q(t_i) \cdot t_i < q(t_i) \cdot a$ for all $a \in \{x_i, z_i, w_i, v_i\}$.

On the other hand, because $v_i > y_i$ and $t_i > y_i$, we have that $q(z_i) \cdot z_i < q(z_i) \cdot a$ for all $a \in \{t_i, v_i\}$, and $q(w_i) \cdot w_i < q(w_i) \cdot a$ for all $a \in \{t_i, v_i\}$.

So far we have shown that

(i) the set $Y_i^2 := \{x_i, t_i, v_i\}$ satisfies the supporting condition with respect to $\{C(x_i, R_i^0), q(t_i), q(v_i)\}$.

(ii) the set $Y_i^3 := \{z_i, w_i, t_i, v_i\}$ satisfies the supporting condition with respect to $\{q(z_i), q(w_i), q(t_i), q(v_i)\}$.

Let $v := (v_i)_{i \in N}$ and $t := (t_i)_{i \in N}$. Let $Y^2 := \{x, t, v\}$ and $Y^3 := \{z, w, t, v\}$. By Lemma A.2, there exist a local dictator $i_1 \in N$ for $\bar{R}$ over $(Y^2, \prod_{i \in N} \mathcal{R}(Y_i^2, \{C(x_i, R_i^0), q(t_i),$

$q(v_i)\}))$, and a local dictator $i_2 \in N$ for $\bar{R}$ over $(Y^3, \prod_{i\in N} \mathcal{R}(Y_i^3, \{q(z_i), q(w_i), q(t_i), q(v_i)\}))$. Recall that agent $i_0 \in N$ is the local dictator for $\bar{R}$ over $(Y^1, \prod_{i\in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\}))$. Let $\mathbf{R}^1 \in \mathcal{R}^n$ be a preference profile such that for all $i \in N$, $C(x_i, R_i^1) = C(x_i, R_i^0)$, and for all $a_i \in \{t_i, v_i, w_i, y_i, z_i\}$, $C(a_i, R_i^1) = \{q(a_i)\}$, and such that

$$x_{i_0} P_{i_0}^1 z_{i_0} P_{i_0}^1 w_{i_0} P_{i_0}^1 t_{i_0} P_{i_0}^1 v_{i_0} P_{i_0}^1 y_{i_0}$$

and, for all $i \in N$ with $i \neq i_0$,

$$x_i P_i^1 v_i P_i^1 t_i P_i^1 w_i P_i^1 z_i P_i^1 y_i.$$

Since $\mathbf{R}^1 \in \prod_{i\in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\})$, and agent i_0 is the local dictator for $\bar{R}$ over $(Y^1, \prod_{i\in N} \mathcal{R}(Y_i^1, \{q(z_i), q(w_i), q(y_i)\}))$, we have $z\bar{P}(\mathbf{R}^1)w$. Because $\mathbf{R}^1 \in \prod_{i\in N} \mathcal{R}(Y_i^3, \{q(z_i), q(w_i), q(t_i), q(v_i)\})$, this implies that $i_0 = i_2$. Hence, we have $t\bar{P}(\mathbf{R}^1)v$. Since $\mathbf{R}^2 \in \prod_{i\in N} \mathcal{R}(Y_i^2, \{C(x_i, R_i^0), q(t_i), q(v_i)\})$, it follows that $i_0 = i_1$.

Let $\mathbf{R}^2 \in \mathcal{R}^n$ be a preference profile such that $x_{i_0} P_{i_0}^2 v_{i_0}$ and, for all $i \in N$, $R_i^2|_{\{x_i, y_i\}} = R_i^0|_{\{x_i, y_i\}}$, and $C(x_i, R_i^2) = C(x_i, R_i^0)$, $C(t_i, R_i^2) = \{q(t_i)\}$, $C(v_i, R_i^2) = \{q(v_i)\}$ and $C(y_i, R_i^2) = C(y_i, R_i^0)$. Since agent $i_0 \in N$ is the local dictator for $\bar{R}$ over $(Y^2, \prod_{i\in N} \mathcal{R}(Y_i^2, \{C(x_i, R_i^0), q(t_i), q(v_i)\}))$ and $\mathbf{R}^2 \in \prod_{i\in N} \mathcal{R}(Y_i^2, \{C(x_i, R_i^0), q(t_i), q(v_i)\})$, we have that $x\bar{P}(\mathbf{R}^2)v$. Recall that, for all $i \in N$, $v_i > y_i$. Hence, by strict monotonicity of preferences, $v_i P_i^2 y_i$ for all $i \in N$. Because the SOF $\bar{R}$ satisfies Weak Pareto, we have $v\bar{P}(\mathbf{R}^2)y$. By transitivity of $\bar{R}$, $x\bar{P}(\mathbf{R}^2)y$. However, since $\bar{R}$ satisfies IIA-MRS, and $C(x_i, R_i^2) = C(x_i, R_i^0)$, $C(y_i, R_i^2) = C(y_i, R_i^0)$, and $y\bar{R}(\mathbf{R}^0)x$, we must have $y\bar{R}(\mathbf{R}^2)x$. This is a contradiction. ■

Lemma A.4 *Let $\bar{R}$ be a SOF satisfying Weak Pareto and IIA-MRS. If $\{x, y, z\} \subset X$ is a free triple, then there exists a local dictator for $\bar{R}$ over $(\{x, y, z\}, \mathcal{R}^n)$.*

Proof. By Lemma A.3, there exist a local dictator i_0 over $(\{x, y\}, \mathcal{R}^n)$, a local dictator i_1 over $(\{y, z\}, \mathcal{R}^n)$, and a local dictator i_2 over $(\{x, z\}, \mathcal{R}^n)$. Suppose that $i_0 \neq i_1$. Let $\mathbf{R} \in \mathcal{R}^n$ be a preference profile such that $x_{i_0} P_{i_0} y_{i_0}$, $y_{i_1} P_{i_1} z_{i_1}$, and $z_{i_2} P_{i_2} x_{i_2}$. Then, we have $x\bar{P}(\mathbf{R})y\bar{P}(\mathbf{R})z\bar{P}(\mathbf{R})x$, which contradicts the transitivity of $\bar{R}(\mathbf{R})$. Hence, we must have $i_0 = i_1$. By the same argument, we have $i_0 = i_1 = i_2$. ■

Proof of Proposition 2: Let $\bar{R}$ be a SOF satisfying Weak Pareto and IIA-MRS. By Lemma A.3, for every free pair $\{x, y\} \subset X$, there exists a local dictator over $(\{x, y\}, \mathcal{R}^n)$. By Lemma A.4 and Bordes and Le Breton [3, Theorem 2], these dictators must be the same individual. Denote this individual by i_0. It remains to show that, for any pair $\{x, y\}$ that is not free, i_0 is the local dictator over $(\{x, y\}, \mathcal{R}^n)$. Suppose, on the contrary, that there exist $\{x, y\} \subset X$ and $\mathbf{R} \in \mathcal{R}^n$ such that $\{x, y\}$ is not a free pair, and $x_{i_0} P_{i_0} y_{i_0}$ but $y\bar{R}(\mathbf{R})x$. Define $z_{i_0} \in \mathbb{R}_+^\ell$ as follows.

Case 1: $\{x, y\}$ is a free pair for i_0.

For all $\lambda \in]0, 1[$, $\{\lambda x + (1-\lambda)y, x\}$ and $\{\lambda x + (1-\lambda)y, y\}$ are free pairs for i_0. By continuity, there exists λ^* such that $x_{i_0} P_{i_0} [\lambda^* x_{i_0} + (1-\lambda^*)y_{i_0}] P_{i_0} y_{i_0}$. Then, let $z_{i_0} := \lambda^* x_{i_0} + (1-\lambda^*)y_{i_0}$.

Case 2: $\{x, y\}$ is not a free pair for i_0.

Then, for all $k \in \{1, \ldots, \ell\}$, $x_{i_0k} \geq y_{i_0k}$ with at least one strict inequality. Note that $y \neq 0$.

Case 2-1: There exists k' such that for all $k \in \{1, \ldots, \ell\}$ with $k \neq k'$, $x_{i_0k} = y_{i_0k}$ and $y_{i_0k'} > 0$.

Then, $x_{i_0k'} > y_{i_0k'} > 0$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathbb{R}^\ell_+$ as $w_{i_0k'} := y_{i_0k'}$ and, for all $k \neq k', w_{i_0k} := y_{i_0k} + \varepsilon$. For sufficiently small ε, we have $x_{i_0}P_{i_0}w_{i_0}P_{i_0}y_{i_0}$ by continuity and strict monotonicity of preferences. Given $\delta > 0$, define $t_{i_0} \in \mathbb{R}^\ell_+$ as $t_{i_0k'} := w_{i_0k'} - \delta$ and, for all $k \neq k', t_{i_0k} := w_{i_0k}$. For sufficiently small δ, we have $x_{i_0}P_{i_0}t_{i_0}P_{i_0}y_{i_0}$, again by continuity and strict monotonicity of preferences. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} := t_{i_0}$.

Case 2-2: There exists k' such that, for all $k \in \{1, \ldots, \ell\}$ with $k \neq k'$, $x_{i_0k} = y_{i_0k}$ and $y_{i_0k'} = 0$.

Then, for all $k \in \{1, \ldots, \ell\}$ with $k \neq k'$, $x_{i_0k} = y_{i_0k} > 0$. Let $k'' \neq k'$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathbb{R}^\ell_+$ as $w_{i_0k''} := x_{i_0k''} - \varepsilon$ and, for all $k \neq k'', w_{i_0k} := x_{i_0k}$. For sufficiently small ε, we have $x_{i_0}P_{i_0}w_{i_0}P_{i_0}y_{i_0}$. Given $\delta > 0$, define $t_{i_0} \in \mathbb{R}^\ell_+$ as $t_{i_0k'} := w_{i_0k'} + \delta$ and, for all $k \neq k', t_{i_0k} := w_{i_0k}$. For sufficiently small δ, we have $x_{i_0}P_{i_0}t_{i_0}P_{i_0}y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} := t_{i_0}$.

Case 2-3: There exist $k', k'' \in \{1, \ldots, \ell\}$ with $k' \neq k''$, $x_{i_0k'} > y_{i_0k'}$ and $x_{i_0k''} > y_{i_0k''}$.

Let k^* be such that $y_{i_0k^*} > 0$. Given $\varepsilon > 0$, define $w_{i_0} \in \mathbb{R}^\ell_+$ as $w_{i_0k^*} := y_{i_0k^*} - \varepsilon$ and, for all $k \neq k^*, w_{i_0k} := x_{i_0k}$. For sufficiently small ε, we have $x_{i_0}P_{i_0}w_{i_0}P_{i_0}y_{i_0}$. Let $k^{**} \neq k^*$. Given $\delta > 0$, define $t_{i_0} \in \mathbb{R}^\ell_+$ as $t_{i_0k^{**}} := w_{i_0k^{**}} + \delta$ and, for all $k \neq k^{**}, t_{i_0k} := w_{i_0k}$. For sufficiently small δ, we have $x_{i_0}P_{i_0}t_{i_0}P_{i_0}y_{i_0}$. Moreover, $\{t, x\}$ and $\{t, y\}$ are free pairs for i_0. Then, let $z_{i_0} = t_{i_0}$.

Next, for each $i \neq i_0$, let $z_i \in \mathbb{R}^\ell_+$ be such that $\{z, x\}$ and $\{z, y\}$ are free pairs for i. By the same construction as above, we can find such $z_i \in \mathbb{R}^\ell_+$ for each i. Let $z = (z_i)_{i \in N} \in \mathbb{R}^{n\ell}_+$. Since i_0 is the dictator over all free pairs, we have that $x\bar{P}(\mathbf{R})z$ and $z\bar{P}(\mathbf{R})y$. By transitivity of $\bar{R}$, we have $x\bar{P}(\mathbf{R})y$, which contradicts the supposition that $y\bar{R}(\mathbf{R})x$. ■

A.2 Proof of Proposition 3

In order to prove the impossibility part, it is convenient to consider various possible sizes of the population. Let $\varepsilon > 0$ be given. Suppose, to the contrary, that there exists a SOF $\bar{R}$ that satisfies Weak Pareto, IIA-ISεN and Anonymity.

Case $n = 2$. Consider the consumption bundles $x := (10\varepsilon, \varepsilon, 0, ...)$, $y := (20\varepsilon, \varepsilon, 0, ...)$, $z := (\varepsilon, 20\varepsilon, 0, ...)$, $w := (\varepsilon, 10\varepsilon, 0, ...)$. Define preference relations $R_1 \in \mathcal{R}$ and $R_2 \in \mathcal{R}$ as follows.

(i) On the subset

$$S_1 := \{v \in R^\ell_+ \mid \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_2 \leq \min\{v_1, 2\varepsilon\}\}$$

we have

$$vR_1v' \Leftrightarrow v_1 + 2v_2 \geq v'_1 + 2v'_2,$$

and on the subset

$$S_2 := \{v \in R^\ell_+ \mid \forall i \in \{3, ..., \ell\}, v_i = 0 \text{ and } v_1 \leq \min\{v_2, 2\varepsilon\}\},$$

we have

$$vR_1v' \Leftrightarrow 2v_1 + v_2 \geq 2v_1' + v_2'.$$

(ii) On $B_\varepsilon(x) \cup B_\varepsilon(y)$,

$$vR_1v' \Leftrightarrow v_1 + 2v_2 + \sum_{k=3}^{\ell} v_k \geq v_1' + 2v_2' + \sum_{k=3}^{\ell} v_k',$$

and on $B_\varepsilon(z) \cup B_\varepsilon(w)$,

$$vR_1v' \Leftrightarrow 2v_1 + v_2 + \sum_{k=3}^{\ell} v_k \geq 2v_1' + v_2' + \sum_{k=3}^{\ell} v_k'.$$

(iii) Note that the projection of $B_\varepsilon(x) \cup B_\varepsilon(y)$ on the subspace of good 1 and good 2, namely, $[B_\varepsilon(x) \cup B_\varepsilon(y)] \cap \{v \in R_+^\ell | \forall i \in \{3, ..., \ell\}, v_i = 0\}$, is included in S_1, and the projection of $B_\varepsilon(z) \cup B_\varepsilon(w)$ on the subspace of good 1 and good 2 is included in S_2.

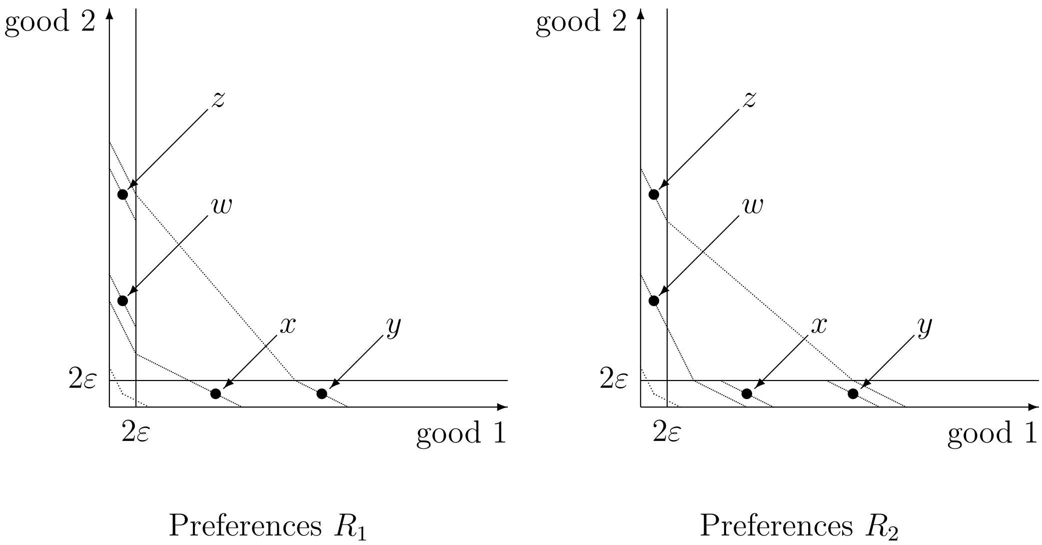

Figure 4: Proof of Proposition 3

Since

$$(w_1 + \varepsilon) + 2(w_2 - 2\varepsilon) > x_1 + 2x_2$$

and

$$2(y_1 - 2\varepsilon) + (y_2 + \varepsilon) > 2z_1 + z_2,$$

it is possible to complete the definition of R_1 so that wP_1x and yP_1z. Then, define R_2 so that it coincides with R_1 on S_1, on S_2, and on $B_\varepsilon(a)$ for all $a \in \{x, y, z, w\}$. Similarly, it is possible to complete the definition of R_2 so that xP_2w and zP_2y. Figure 4 illustrates this construction.

If the profile of preferences is $\mathbf{R} := (R_1, R_2)$, by Weak Pareto we have

$$(y, x)\bar{P}(\mathbf{R})(z, w) \text{ and } (w, z)\bar{P}(\mathbf{R})(x, y).$$

If the profile of preferences is $\mathbf{R}' := (R_1, R_1)$, by Anonymity we have

$$(y, x)\bar{I}(\mathbf{R}')(x, y) \text{ and } (w, z)\bar{I}(\mathbf{R}')(z, w).$$

Since R_1 and R_2 coincide on $B_\varepsilon(a)$ for all $a \in \{x, y, z, w\}$, it follows from IIA-ISεN that

$$\begin{aligned} (y, x)\bar{I}(\mathbf{R}')(x, y) &\Leftrightarrow (y, x)\bar{I}(\mathbf{R})(x, y), \\ (w, z)\bar{I}(\mathbf{R}')(z, w) &\Leftrightarrow (w, z)\bar{I}(\mathbf{R})(z, w). \end{aligned}$$

By transitivity, $(x, y)\bar{P}(\mathbf{R})(x, y)$, which is impossible.

Case $n = 3$. Consider the consumption bundles $x := (10\varepsilon, \frac{2\varepsilon}{3}, 0, ...)$, $y := (20\varepsilon, \frac{2\varepsilon}{3}, 0, ...)$, $t := (15\varepsilon, \frac{2\varepsilon}{3}, 0, ...)$, $z := (\frac{2\varepsilon}{3}, 20\varepsilon, 0, ...)$, $w := (\frac{2\varepsilon}{3}, 10\varepsilon, 0, ...)$, $r := (\frac{2\varepsilon}{3}, 15\varepsilon, 0, ...)$. Define preference relations R_1, R_2 and R_3 as above on the subset S_1, on S_2, and on $B_\varepsilon(a)$ for all $a \in \{x, y, z, w, t, r\}$. Complete their definitions so that yP_1z, wP_1x, tP_2r, zP_2y, xP_3w, and rP_3t.

If the profile of preferences is $\mathbf{R} := (R_1, R_2, R_3)$, then by Weak Pareto we have

$$(y, t, x)\bar{P}(\mathbf{R})(z, r, w) \text{ and } (w, z, r)\bar{P}(\mathbf{R})(x, y, t).$$

If the profile of preferences is $\mathbf{R}' := (R_1, R_1, R_1)$, by Anonymity we have

$$(y, t, x)\bar{I}(\mathbf{R}')(x, y, t) \text{ and } (w, z, r)\bar{I}(\mathbf{R}')(z, r, w).$$

Since R_1, R_2 and R_3 coincide on $B_\varepsilon(a)$ for all $a \in \{x, y, t, z, w, r\}$, it follows from IIA-IS$\varepsilon$N that

$$\begin{aligned} (y, t, x)\bar{I}(\mathbf{R}')(x, y, t) &\Leftrightarrow (y, t, x)\bar{I}(\mathbf{R})(x, y, t), \\ (w, z, r)\bar{I}(\mathbf{R}')(z, r, w) &\Leftrightarrow (w, z, r)\bar{I}(\mathbf{R})(z, r, w). \end{aligned}$$

By transitivity, $(x, y, t)\bar{P}(\mathbf{R})(x, y, t)$, which is impossible.

Case $n = 2k$. Partition the population into k pairs, and construct an argument similar to the case $n = 2$, with the consumption bundles $x = (10\varepsilon, \frac{2\varepsilon}{n}, 0, ...)$, $y = (20\varepsilon, \frac{2\varepsilon}{n}, 0, ...)$, $z = (\frac{2\varepsilon}{n}, 20\varepsilon, 0, ...)$, $w = (\frac{2\varepsilon}{n}, 10\varepsilon, 0, ...)$, and the allocations $(y, x, y, x, ...)$, $(x, y, x, y, ...)$, $(z, w, z, w, ...)$ and $(w, z, w, z, ...)$.

Case $n = 2k + 1$. Partition the population into $k - 1$ pairs and one triple, and construct an argument combining the cases $n = 2$ and 3, with the consumption bundles $x = (10\varepsilon, \frac{2\varepsilon}{n}, 0, ...)$, $y = (20\varepsilon, \frac{2\varepsilon}{n}, 0, ...)$, $t = (15\varepsilon, \frac{2\varepsilon}{n}, 0, ...)$, $z = (\frac{2\varepsilon}{n}, 20\varepsilon, 0, ...)$, $w =$

$(\frac{2\varepsilon}{n}, 10\varepsilon, 0, ...)$, $r = (\frac{2\varepsilon}{n}, 15\varepsilon, 0, ...)$, and the allocations $(y, x, y, x, ..., y, t, x)$, $(x, y, x, y, ..., x, y, t)$, $(z, w, z, w, ...z, r, w)$ and $(w, z, w, z, ..., w, z, r)$. ■

Remark: One may consider a condition which is weaker than both IIA–ISεN and IIA-ISFA by allowing the social ranking of any two allocations x, y to depend on the portions of indifference surfaces in the union of the ε-neighborhoods of x_i and y_i, and the set $\Omega(x, y)$. With this weaker condition, the same impossibility still holds as in Proposition 3. The above proof can be applied without any changes to derive this result.

A.3 Proof of Lemma 1

To prove Lemma 1, we need an auxiliary lemma. Define

$$\begin{aligned} X_1 &:= \{x_i \in \mathbb{R}^\ell_+ \setminus \{0\} \mid \forall k \geq 2, x_{ik} = 0\} \\ X_2 &:= \{x_i \in \mathbb{R}^\ell_+ \setminus \{0\} \mid \forall k \neq 2, x_{ik} = 0\}. \end{aligned}$$

Lemma A.5 *For all $R_i \in \mathcal{R}$, and all $x, y \in X$, there exists $R_i^* \in \mathcal{R}$ such that*

$$\begin{aligned} I(x_i, R_i) \cap \Omega(x, y) &= I(x_i, R_i^*) \cap \Omega(x, y) \\ I(y_i, R_i) \cap \Omega(x, y) &= I(y_i, R_i^*) \cap \Omega(x, y) \\ I(x_i, R_i^*) \cap X_1 &\neq \emptyset \\ I(y_i, R_i^*) \cap X_1 &\neq \emptyset \end{aligned}$$

Proof. Let $R_i \in \mathcal{R}$ and $x, y \in X$ be given. Without loss of generality, assume that $y_i R_i x_i$. Define $A := I(x_i, R_i) \cap \Omega(x, y)$ and

$$U(x_i, R_i^*) := \bigcap_{a \in A} \left[\bigcap_{q \in C(a, R_i)} H(a, q) \right]$$

where we recall that $H(a, q) = \{b \in \mathbb{R}^\ell_+ \mid q \cdot b \geq q \cdot a\}$. Let $I(x_i, R_i^*)$ be the boundary of $U(x_i, R_i^*)$.

Define a function $g : A \to \mathbb{R}_+$ as follows: For every $a \in A$, if $(a_1 + 1, 0, \ldots, 0) P_i a$, then let $g(a) = 0$, and otherwise, let $g(a) \in \mathbb{R}$ be such that $(a_1 + 1, g(a)a_2, \ldots, g(a)a_\ell) I_i a$. By continuity and strict monotonicity of R_i, $g(a)$ exists uniquely and $0 \leq g(a) < 1$. By continuity of R_i, g is continuous. For every $a \in A$, let $b(a) := (a_1 + 1, g(a)a_2, \ldots, g(a)a_\ell)$. Define $f : A \to X_1$ by

$$\begin{aligned} f(a) &:= a + \frac{1}{1 - g(a)} [b(a) - a] \\ &= \left(a_1 + \frac{1}{1 - g(a)}, 0, \ldots, 0 \right). \end{aligned}$$

Since $b(a) R_i a$, it follows that for every $q \in C(a, R_i)$, $q \cdot b(a) \geq q \cdot a$, and so $q \cdot f(a) \geq q \cdot a$. Hence, $f(a) \in H(a, q)$.

The function f is continuous, and the set A is compact and nonempty. Hence, the set $f(A)$ is compact and nonempty. Therefore, there exists $a^* \in A$ such that $||f(a^*)|| = \max_{a\in A} ||f(a)|| = \max_{a\in A} \left[a_1 + \frac{1}{1-g(a)}\right]$. Then, for all $a \in A$, and all $q \in C(a, R_i)$, since $f(a) \in H(a, q)$ and $f(a^*) \geq f(a)$, we have $f(a^*) \in H(a, q)$. Thus, $f(a^*) \in U(x_i, R_i^*)$, which proves that $U(x_i, R_i^*) \cap X_1 \neq \emptyset$. By continuity and strict monotonicity of preferences, $I(x_i, R_i^*) \cap X_1 \neq \emptyset$.

If $y_i I_i x_i$, then we are done. Assume that $y_i P_i x_i$. Define

$$U(y_i, R_i^*) := \bigcap_{a\in I(y_i,R_i)\cap\Omega(x,y)} \left[\bigcap_{q\in C(a,R_i)} H(a, q)\right].$$

By continuity of preferences, there exists $\delta > 0$ such that for all $z_i \in [(1 + \delta)I_i(x_i, R_i)] \cap \Omega(x, y)$, $y_i P_i z_i$. Define

$$\tilde{U}(y_i, R_i^*) := U(y_i, R_i^*) \cap (1 + \delta)U(x_i, R_i^*).$$

Then, let $I(y_i, R_i^*)$ be the boundary of $\tilde{U}(y_i, R_i^*)$. Note that $I(x_i, R_i^*) \cap I(y_i, R_i^*) = \emptyset$. A similar argument as above shows that $U(y_i, R_i^*) \cap X_1 \neq \emptyset$. Since $U(x_i, R_i^*) \cap X_1 \neq \emptyset$, we have $[(1 + \delta)U(x_i, R_i^*)] \cap X_1 \neq \emptyset$. Thus, $\tilde{U}(y_i, R_i^*) \cap X_1 \neq \emptyset$. By continuity and strict monotonicity of preferences, $I(y_i, R_i^*) \cap X_1 \neq \emptyset$. ■

Proof of Lemma 1.
Let $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^n$, $x, y \in X$ be such that for all $i \in N$, R_i and R_i' agree on $\{x_i, y_i\}$, and for no $i \in N$, $x_i I_i y_i$. Assume that $x\bar{P}(\mathbf{R})y$.

By Lemma A.5, there exists $\mathbf{R}^* \in \mathcal{R}^n$ such that for all $i \in N$,

$$\begin{aligned} I(x_i, R_i) \cap \Omega(x, y) &= I(x_i, R_i^*) \cap \Omega(x, y) \\ I(y_i, R_i) \cap \Omega(x, y) &= I(y_i, R_i^*) \cap \Omega(x, y) \\ I(x_i, R_i^*) \cap X_1 &\neq \emptyset \\ I(y_i, R_i^*) \cap X_1 &\neq \emptyset, \end{aligned}$$

and similarly there exists $\mathbf{R}'^* \in \mathcal{R}^n$ such that for all $i \in N$, ,

$$\begin{aligned} I(x_i, R_i') \cap \Omega(x, y) &= I(x_i, R_i'^*) \cap \Omega(x, y) \\ I(y_i, R_i') \cap \Omega(x, y) &= I(y_i, R_i'^*) \cap \Omega(x, y) \\ I(x_i, R_i'^*) \cap X_2 &\neq \emptyset \\ I(y_i, R_i'^*) \cap X_2 &\neq \emptyset. \end{aligned}$$

By strict monotonicity of preferences, each of $I(x_i, R_i^*) \cap X_1$, $I(y_i, R_i^*) \cap X_1$, $I(x_i, R_i'^*) \cap X_2$, and $I(y_i, R_i'^*) \cap X_2$ is a singleton. Define $x^1, y^1 \in X_1^n$ by $\{x_i^1\} := I(x_i, R_i^*) \cap X_1$ and $\{y_i^1\} := I(y_i, R_i^*) \cap X_1$ for all $i \in N$. Notice that for all $i \in N$, $x_{i1}^1 > 0$, $y_{i1}^1 > 0$ because

$x, y \in X$ and preferences are strictly monotonic. Construct $x^{1*}, y^{1*} \in X_1^n$ as follows: for all $i \in N$,

$$\begin{aligned} x_{i1}^{1*} &:= x_{i1}^1 + \frac{1}{3}\left|x_{i1}^1 - y_{i1}^1\right| \\ y_{i1}^{1*} &:= \max\left\{\frac{1}{2}y_{i1}^1, y_{i1}^1 - \frac{1}{3}\left|x_{i1}^1 - y_{i1}^1\right|\right\}. \end{aligned}$$

Notice that, for all $i \in N$,

$$\begin{aligned} x_{i1}^{1*} &> y_{i1}^{1*} \Leftrightarrow x_i P_i y_i \\ y_{i1}^{1*} &> x_{i1}^{1*} \Leftrightarrow y_i P_i x_i. \end{aligned}$$

By Weak Pareto, $x^{1*}\bar{P}(\mathbf{R}^*)x$ and $y\bar{P}(\mathbf{R}^*)y^{1*}$. By IIA-ISFA, $x\bar{P}(\mathbf{R}^*)y$. Therefore, by transitivity,

$$x^{1*}\bar{P}(\mathbf{R}^*)y^{1*}.$$

Now, define $x^2, y^2 \in X_2^n$ by $\{x_i^2\} := I(x_i, R_i'^*) \cap X_2$ and $\{y_i^2\} := I(y_i, R_i'^*) \cap X_2$ for all $i \in N$. Again, $x_{i2}^2 > 0$, $y_{i2}^2 > 0$ for all $i \in N$. Construct $x^{2*}, y^{2*} \in X_2^n$ as follows: for all $i \in N$,

$$\begin{aligned} x_{i2}^{2*} &:= \max\left\{\frac{1}{2}x_{i2}^2, x_{i2}^2 - \frac{1}{3}\left|x_{i2}^2 - y_{i2}^2\right|\right\} \\ y_{i2}^{2*} &:= y_{i2}^2 + \frac{1}{3}\left|x_{i2}^2 - y_{i2}^2\right|. \end{aligned}$$

Notice that, for all $i \in N$,

$$\begin{aligned} x_{i2}^{2*} &> y_{i2}^{2*} \Leftrightarrow x_i P_i' y_i \Leftrightarrow x_i P_i y_i \Leftrightarrow x_{i1}^{1*} > y_{i1}^{1*} \\ y_{i2}^{2*} &> x_{i2}^{2*} \Leftrightarrow y_i P_i' x_i \Leftrightarrow y_i P_i x_i \Leftrightarrow y_{i1}^{1*} > x_{i1}^{1*}. \end{aligned}$$

By Weak Pareto, $x\bar{P}(\mathbf{R}'^*)x^{2*}$ and $y^{2*}\bar{P}(\mathbf{R}'^*)y$.

Let $\mathbf{R}^{**} \in \mathcal{R}^n$ be such that for all $i \in N$,

$$x_i^{2*} P_i^{**} x_i^{1*} \text{ and } y_i^{1*} P_i^{**} y_i^{2*}.$$

Notice that, for all $i \in N$,

$$\begin{aligned} I(x_i^{1*}, R_i^{**}) \cap \Omega(x^{1*}, y^{1*}) &= I(x_i^{1*}, R_i^*) \cap \Omega(x^{1*}, y^{1*}) = \{x_i^{1*}\}, \\ I(y_i^{1*}, R_i^{**}) \cap \Omega(x^{1*}, y^{1*}) &= I(y_i^{1*}, R_i^*) \cap \Omega(x^{1*}, y^{1*}) = \{y_i^{1*}\}. \end{aligned}$$

Therefore, by IIA-ISFA, $x^{1*}\bar{P}(\mathbf{R}^{**})y^{1*}$. By Weak Pareto, $x^{2*}\bar{P}(\mathbf{R}^{**})x^{1*}$ and $y^{1*}\bar{P}(\mathbf{R}^{**})y^{2*}$, so that by transitivity, $x^{2*}\bar{P}(\mathbf{R}^{**})y^{2*}$.

Now, we also have that, for all $i \in N$,

$$\begin{aligned} I(x_i^{2*}, R_i^{**}) \cap \Omega(x^{2*}, y^{2*}) &= I(x_i^{2*}, R_i'^*) \cap \Omega(x^{2*}, y^{2*}) = \{x_i^{2*}\}, \\ I(y_i^{2*}, R_i^{**}) \cap \Omega(x^{2*}, y^{2*}) &= I(y_i^{2*}, R_i'^*) \cap \Omega(x^{2*}, y^{2*}) = \{y_i^{2*}\}. \end{aligned}$$

By IIA-ISFA again, $x^{2*}\bar{P}(\mathbf{R}'^*)y^{2*}$. By transitivity, we deduce $x\bar{P}(\mathbf{R}'^*)y$. Finally, by IIA-ISFA,

$$x\bar{P}(\mathbf{R}')y.$$

We have proved that $x\bar{P}(\mathbf{R})y$ implies $x\bar{P}(\mathbf{R}')y$,. It follows from symmetry of the argument that $y\bar{P}(\mathbf{R})x$ implies $y\bar{P}(\mathbf{R}')x$, and that $x\bar{I}(\mathbf{R})y$ implies $x\bar{I}(\mathbf{R}')y$. ■

A.4 Logical Relations between the IIA Axioms

1. IIA-MRS implies neither IIA-ISFA nor IIA.
Consider the following SOF: $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, $x\ \bar{R}(\mathbf{R})\ y$ if and only if
(i) $\exists p \in \mathbb{R}_+^\ell$ such that $\forall i \in N$, $p \in C(x_i, R_i)$, and $\forall i, j \in N$, $p \cdot x_i = p \cdot x_j$, or
(ii) $\nexists p \in \mathbb{R}_+^\ell$ such that $\forall i \in N$, $p \in C(y_i, R_i)$, and $\forall i, j \in N$, $p \cdot y_i = p \cdot y_j$.
This SOF satisfies IIA-MRS but violates IIA-ISFA and hence IIA.

2. IIA-ISεN implies neither IIA-ISFA nor IIA.
This is derived from fact 1 above and the fact that IIA-MRS implies IIA-ISεN.

3. IIA-ISεN does not imply IIA-MRS.
This is derived from Propositions 2 and 3.

4. IIA-WIS implies neither IIA-ISεN nor IIA-ISFA.
This is derived from Propositions 3, 4 and Corollary 1.

5. IIA-ISFA implies none of IIA-ISεN, IIA-MRS and IIA.
Fix $\omega_0 := (1, \ldots, 1) \in \mathbb{R}^\ell$. For each $i \in N$, each $R_i \in \mathcal{R}$ and each $x_i \in \mathbb{R}_+^\ell$, let $\alpha(x_i, R_i) \in \mathbb{R}_+$ be defined as in the proof of Proposition 5. Consider the following SOF: $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, $x\ \bar{R}(\mathbf{R})\ y$ if and only if (i) $\nexists \lambda \in \mathbb{R}_+$ such that $\sum_{i \in N} y_i = \lambda\omega_0$, or (ii) $\exists \lambda, \lambda' \in \mathbb{R}_+$ such that $\sum_{i \in N} x_i = \lambda\omega_0$ and $\sum_{i \in N} y_i = \lambda'\omega_0$, and $\min_{i \in N} \alpha(x_i, R_i) \geq \min_{i \in N} \alpha(y_i, R_i)$. This SOF satisfies IIA-ISFA, but it violates IIA-ISεN and hence IIA-MRS and IIA.

6. IIA-ISPω_0 implies none of IIA-ISεN, IIA-MRS, IIA-ISFA and IIA.
This is derived from Propositions 3-5.

7. IIA-WIS does not imply IIA-ISPω_0.
Fix $p := (1, \ldots, 1) \in \mathbb{R}^\ell$. For each $i \in N$, each $R_i \in \mathcal{R}$ and each $x_i \in \mathbb{R}_+^\ell$, define $e(x_i, R_i) := \min\{p \cdot y_i | y_i \in I(x_i, R_i)\}$. (That is, $e(x_i, R_i)$ is the minimum expenditure to attain $I(x_i, R_i)$ at p.) Consider the following SOF: $\forall \mathbf{R} \in \mathcal{R}^n$, $\forall x, y \in \mathbb{R}_+^{n\ell}$, $x\bar{R}(\mathbf{R})y$ if and only if $\min_{i \in N} e(x_i, R_i) \geq \min_{i \in N} e(y_i, R_i)$. This SOF satisfies IIA-WIS but violates IIA-ISPω_0.

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